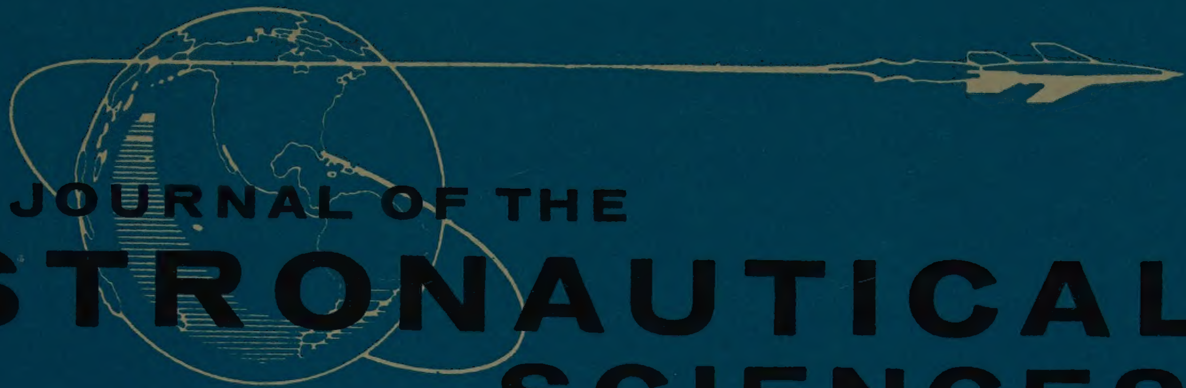
THE JOURNAL OF THE
ASTRONAUTICAL SCIENCES



VOLUME VII, NO. 2 SUMMER 1960

CONTENTS

PUBLISHED BY THE AMERICAN ASTRONAUTICAL SOCIETY, INC.

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Journal of the Astronautical Sciences

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Published quarterly by the American A tronautical Society, Inc. at 428 E. Prest Street, Baltimore 2, Maryland

Address all Journal correspondence to B 24721, Los Angeles 24, Calif.

Subscription Rates: One year $5.00; forei $6.00; single copy $1.25. The Journal is pu lished quarterly. Members of the Socie receive the Journal as a part of their du

Application to mail at second class pos age rates is pending at Baltimore, Mar land.

Limitations of a Rocket Propulsion System with Variable Exhaust Velocity taking into Consideration Nuclear Data*

Trutz Foelsche†

Abstract

It is shown that, for nuclear physical reasons, an end velocity higher than $0.134c$ cannot be attained by a rocket using a propulsion system which maximizes the end velocity, at given energy content and end mass, by varying the exhaust velocity or by using propellants with variable combustion heat.

Limiting Factors for the End Velocity of a Rocket

In discussions about reaching high velocities with a rocket and especially reaching relativistic velocities [1]–[6] [10] [12], two limiting factors are outstandingly apparent.

One of these is that the mass conversion factor g‡ of nuclear fuel, (that fraction of fuel mass which can be converted into energy) is low. As a consequence unrealistic values of initial mass to final mass would be required to obtain such high end velocities.

The other is that the specific power of the rocket engine and the burning time are limited. This second factor can be expressed in the form: the energy E for a given end mass M_e of the rocket, or the product

$$E/M_e = P/M_e \cdot t \text{ is limited,}$$

where P/M_e is the power engine mass and t the burning time. (Here we equalize the end mass and the engine mass, that means we include the payload and structure mass in the engine mass for simplicity.)

The end velocity which cannot be surpassed is determined by E/M_e or by this product $(P/M_e)t$ as may be seen from the formula

$$M_e \frac{u_e^2}{2} \leqq E,$$

where u_e is the end velocity. The "characteristic velocity $v_c = \sqrt{2E/M_e}$" obtained in transforming the whole energy into kinetic energy of the end mass cannot be surpassed.

To give an example of how our present technology compares with the task of reaching relativistic velocities, it may be mentioned that a burning time of 50 years would be required, using an optimal propulsion system and an engine having the specific power of the V_2 (here assumed to be 500 kw/kg engine mass) in order to attain an end velocity of only $0.134c$. Such an optimum propulsion system is defined in the following discussion.

An Optimum Propulsion System with Variable Exhaust Velocity

Propulsion systems with constant exhaust velocity c_p* ($c_p = c_{\text{propellant}}$, relative to the vehicle) are not the most favorable systems with respect to end velocity, if energy E and end mass M_e are given. An optimum propulsion system can be visualized, where the exhaust velocity c_p as a function of the decreasing mass of the rocket is varied in such a way that the end velocity u_e of the rocket will be a maximum [7]–[11]. Using classical mechanics the exhaust velocity must be inversely proportional to the instantaneous mass of the rocket. The relativistic treatment of the problem was given by H. Knothe [10] using a general inequality, which contains Schwartz's inequality as a special case. The above classical approximation for the dependence of c_p on the rocket mass is a good approximation at the relativistically low velocities considered here. We may summarize some of the qualities of this propulsion system [6b].

The end velocity u_e which can be approached by increasing the mass ratio M_a/M_e (M_a initial mass of the rocket) is in classical approximation $u_e = \sqrt{2E/M_e}$; this means there is nearly no loss of energy in the exhausted jet. A predetermined end velocity u_e is reached with a minimum of energy, that is $E = M_e(u_e^2/2)$, if $M_a/M_e \gg 1$. Thus the optimum system has the advantage that the product of specific power of the engine and burning time for attaining the predetermined end velocity will be a minimum since this product $(P/M_e)t$ is proportional to the total energy $E = P \cdot t$. Assuming a constant power output of the propulsion unit the acceleration will be constant; however this is not true of the thrust. Thus the optimum system avoids the high acceleration of the $c_p =$ constant system at the end of the propulsion phase.

Maximal Final Velocity of the Optimum System in Comparison to that of the System having Constant

* Adapted from Colloquim notes 20 Jan. 1958, AFMDC, Holloman A. F. B.

† N.A.S.A. Langley Research Center.

‡ g has no relationship to the acceleration in the gravitational field of the earth for which normally the same letter is used.

* The reader is reminded that c_p is the exhaust velocity and is not the specific heat at constant pressure which usually carries the same symbol.

Exhaust Velocity if Specific Power x Burning Time is Given

For quantitative comparison with a propulsion system having constant exhaust velocity the classical approximations for u_e may be recalled [10]. If c_p is any function of the variable mass M of the rocket, the Schwartz's inequality can be applied for comparison of

$$u_e = \int \frac{c_p}{M}\, dM \tag{1}$$

and

$$E = \int \frac{c_p^2}{2}\, dM \tag{2}$$

as follows:

$$u_e^2 = \left(\int \frac{c_p}{M}\, dM\right)^2 \leq \int c_p^2\, dM \int \frac{1}{M^2}\, dM = \underbrace{\int c_p^2\, dM}_{2E} \cdot \left(\frac{1}{M_e} - \frac{1}{M_a}\right)$$

The equality (Optimum System) holds only if

$$c_p = \lambda/M$$

where λ is related to E and M_a, M_e by

$$E = \frac{1}{2}\int \frac{\lambda^2}{M^2}\, dM = \frac{\lambda^2}{2}\left(\frac{1}{M_e} - \frac{1}{M_a}\right),$$

$$\gamma = \sqrt{\frac{2E}{\dfrac{1}{M_e} - \dfrac{1}{M_a}}}$$

The combustion heats $H(M)$ of the variable fuel have to be

$$H = \frac{c_p^2}{2} = \bar{H}\,\frac{M_e M_a}{M^2} \tag{3}$$

where the average combustion heat $\bar{H}$ is defined by $E = \bar{H}(M_a - M_e)$. Hence from (1) it follows that for the optimum system ($c_p \sim$ variable):

$$\begin{aligned} u_e = \int \frac{\lambda}{M^2}\, dM &= \sqrt{2E\left(\frac{1}{M_e} - \frac{1}{M_a}\right)} \\ &= \sqrt{2\bar{H}}\sqrt{\left(\frac{M_a}{M_e} - 1\right)\left(1 - \frac{M_e}{M_a}\right)} \\ &= \sqrt{\frac{2E}{M_e}}\sqrt{1 - \frac{M_e}{M_a}} \end{aligned} \tag{A}$$

Considering the c_p = const. system we have the known equations:

$$H = c_p^2/2 = \text{const.} \qquad E = H(M_a - M_e)$$

$$\begin{aligned} u_e = c_p \ln\frac{M_a}{M_e} &= \sqrt{2H}\ln\frac{M_a}{M_e} \\ &= \sqrt{\frac{2E}{M_e}}\sqrt{\frac{1}{M_a/M_e - 1}}\ln\frac{M_a}{M_e} \end{aligned} \tag{B}$$

If the engine and the mission is power- and tin limited, that means if

$$\frac{E}{M_e} = \frac{P}{M_e}\cdot t \qquad \text{is giv}$$

then a maximum increase of 25 percent for u_e can obtained by using the optimum system, as is fi shown by D. B. Langmuir [8] and T. H. Irving [and can be recognized from (A) and (B) as follow

For the c_p = const. system, we see from (B), varyi the mass ratio, that at a given characteristic veloci $v_c = \sqrt{2E/M_e}$ the end velocity u_e has a maximu value $u_e = 0.8\cdot v_c$ at the mass ratio $M_a/M_e = 5$ a at the exhaust velocity $c_p = 0.5\cdot v_c$.

For the optimum system ($c_p \approx 1/M$), u_e in (approaches its maximum $u_e = \sqrt{2E/M_e}$ as $M_a/$ is increased. Practically the maximum, the ch acteristic velocity itself, is reached at M_a/M_e = For a given characteristic velocity $v_c = 0.134c$ bo the systems are compared in Fig. 1 (dashed line

Comparison of the Optimum Propulsion Sy tems with the System having Consta Exhaust Velocity, if a Low Mass Co version Factor of the Fuel Matter Given

Let us assume, however, that the engine and t mission are practically *not* time limited or that t specific power of advanced engines for long burni times could be essentially increased, so that the produ specific power x burning time yields a characteris velocity essentially larger than 0.134 c.

Then the question arises of how the end veloci of a rocket using such an optimum propulsion syste will be limited by the low mass conversion factor nuclear fuel, that is to say which end velocities can attained with the $c_p \sim 1/M$ system under the co dition that the energy unit is coupled with a gre amount of mass. Most of the work considering re tivistic mechanics and propulsion has dealt with t c_p = const. system [1]–[6a] [12]. The relativistic fo mulas for u_e for such systems are given by Ackeret [Saenger [4] etc. [2]–[6] [10] [12]. From these formu can be calculated the mass conversion factors whi would be needed to reach relativistic velocities, several given mass ratios. Referring to the mass co version factor, it is emphasized in [6a] that there exi an upper limit for the velocity attainable with a c_p const. system, at a few tenths of the velocity of lig using normal matter as fuel. Nature put a basic barr for the conversion of matter into energy, this barr is the stability of protons, or the law of conservati of the number of protons and neutrons during a nuclear process which involves normal matter at te peratures less than 10^{14}°K. Only the mass equivale of the binding energy of the nuclei is available conversion into heat or kinetic energy in the nucl processes of fission and fusion and the mass conversi

actor for any such process cannot surpass the value of $g = \frac{9}{1000}$, as is shown in the experimentally established Binding - Energy - Per - Particle curve (BEPP - Curve). There is seen, at present, no possibility of creating macroscopic amounts of anti-matter on earth and to have a fuel with a higher mass conversion factor then $g = \frac{9}{1000}$, disregarding the problems of controlling the matter and anti-matter reaction and direction of the annihilation radiations.

The upper limit of the end velocity of a rocket using the described optimum propulsion system may be calculated, if no fuel with a mass conversion factor g larger than $\frac{9}{1000}$ is available.*

Thus we have a problem, which differs from that treated in the above section where E/M_e was given. We compare the c_p = const system with the $c_p \sim 1/M$ system if $g_{\max} = H_{\max}/c^2$ is given as the same for both systems, where H is the combustion heat of the fuels.

The optimum propulsion system requires a fuel with variable heat of combustion and this variable range must increase as the assumed mass ratio of the rocket increases. In the classical approach the limits for the required mass conversion factors g of the fuel are (equation 3):

$$g_{\max} = \frac{H_{\max}}{c^2} = \frac{\bar{H}}{c^2} \cdot \frac{M_a}{M_e} \qquad g_{\min} = \frac{\bar{H}}{c^2} \cdot \frac{M_e}{M_a}$$

where $\bar{H}$, the average heat of combustion, is defined by

$$\bar{H} = \bar{g}c^2 = \frac{E}{M_a - M_e} \tag{4}$$

However, the maximum mass conversion factor

$$g_{\max} = \bar{g}\,\frac{M_a}{M_e} = \frac{9}{1000} \tag{5}$$

of normal matter, which is the only possible fuel, cannot be surpassed. Therefore, the average heat of combustion of the fuel will decrease as the assumed mass ratio increases. The fuel degenerates to a certain degree, although the attainable end velocity increases. The energy content of such rockets is limited by the value $c^2 \cdot g_{\max} \cdot M_e$ for large M_a/M_e, as can be seen by substituting $g_{\max}(M_e/M_a)$ for $\bar{g}$ from equation (5) in equation (4). We obtain in equation (4)

$$E = \bar{H}(M_a - M_e) = g_{\max} \cdot c^2 \cdot M_e \left(1 - \frac{M_e}{M_a}\right)$$
$$\approx g_{\max} \cdot c^2 \cdot M_e \text{ for large} \cdot \frac{M_a}{M_e} \tag{6}$$

This energy content cannot be increased by increasing the mass ratio or the mass of the fuel as in the case of propulsion with c_p = const. Therefore, supposing $g_{\max} = \frac{9}{1000}$, the greatest end velocity, that can be approached is given by the formula

$$u_e = \sqrt{2E/M_e} = \sqrt{2c^2 g_{\max}} = 0.134c$$

which results by inserting E from (6), see also equation (A).

For comparing quantitatively the c_p = const and $c_p \sim 1/M$ systems at given $g_{\max}$, we consider again the equations (A) (B) for u_e, introducing the maximum mass conversion factor $g_{\max} = \frac{9}{1000}$ or the maximum combustion heat $H_{\max} = g_{\max} \cdot c^2$.

For the $c_p \sim 1/M$ system we get from (A) and $\bar{H} = g_{\max} \cdot c^2 (M_e/M_a)$

$$u_e = \sqrt{2\bar{H}}\sqrt{\left(\frac{M_a}{M_e} - 1\right)\left(1 - \frac{M_e}{M_a}\right)}$$
$$= \sqrt{2H_{\max}}\left(1 - \frac{M_e}{M_a}\right)$$
$$\to \sqrt{2H_{\max}} \quad \text{for increasing } M_a/M_e.$$

For c_p = const however, we obtain from (B)

$$u_e = \sqrt{2H_{\max}} \ln M_a/M_e,$$

which increases arbitrarily if M_a/M_e increases.

We have thus the surprising result that if $g_{\max}$ is given the c_p = const propulsion system is far superior at large mass ratios in respect to reaching high end velocities. The maximum mass conversion factor $g = \frac{9}{1000}$ restricts the applicability of the optimum method to an end velocity smaller then $0.134c$. Using the optimum propulsion at the beginning, and then continuing the propulsion with $g_{\max}$ = const, a higher mass ratio would be required than with c_p = const propulsion using the best fuel available from the beginning on. In the latter case of course a greater amount of the optimum fuel and of energy would be consumed. In all these considerations we assume that the internal efficiency of the propulsion motor is equal to one, and furthermore, that such an optimum thermo nuclear reaction with $g = \frac{9}{1000}$ could be realized and controlled in a rocket. This problem is yet unsolved even for the fusion reaction $D + T \to He^4 + n$ having the mass conversion factor $g = \frac{4}{1000}$. Using the relativistic kinematic equations only minor corrections will result at these relativistically low velocities of the rocket and propellant.

In Fig. 1 the solid lines show the values u_e/c, which can be attained by varying the mass ratio for both the systems:

$$\left.\begin{aligned} c_p &\sim \text{var}\,\frac{1}{M} \\ c_p &= \text{const} \end{aligned}\right\} \text{ if } g_{\max} = \frac{9}{1000} \text{ is given.}$$

* We consider here only the case of variable c_p treated up to now, that is the case where the propellant is the combustion products of the fuels or where the combustion products having rest mass of a separate energy source are contained in the propellant. The system with variable c_p, where the combustion products of a separate energy source are partially thrown overboard without contributing to the thrust, to reach higher exhaust velocities, is treated in a following note.

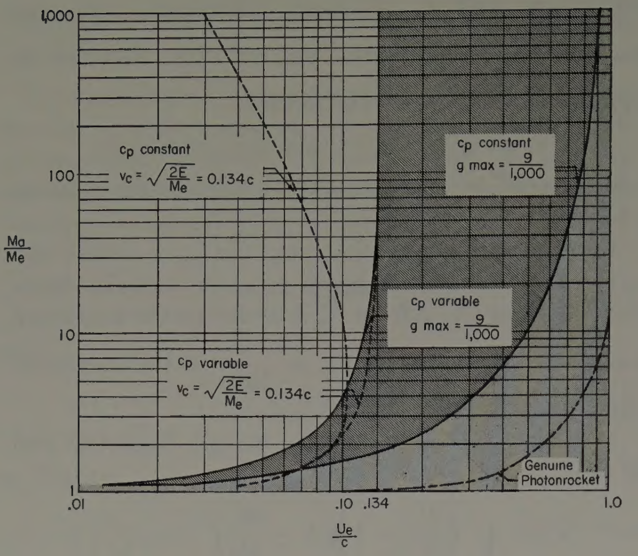

FIG. 1. A comparison of the optimum system (c_p variable) and the constant c_p system with reference to attainable end velocities.

—— For a given maximum mass conversion factor, $g_{\max} = \dfrac{9}{1{,}000}$

--- For a given $\dfrac{E}{M_e} = \dfrac{\text{Power}}{\text{engine mass}} \times$ burning time $= H_{\max}$

$= g_{\max} \times c^2$ which is a given characteristic velocity,

$$v_c = \sqrt{\frac{2E}{M_e}} = 0.134c$$

The shadowed area would be only accessible by $c_p =$ const. The dotted area cannot be reached at all if $g < \dfrac{9}{1000}$. Especially a genuine photon rocket would presuppose a mass conversion factor one.

The dashed lines show the upper limits of end velocities which could be attained, if $E/M_e = (P/M_e)t = g_{\max} \cdot c^2 = 8.1 \times 10^{18}$ erg/gr, which corresponds to a characteristic velocity $0.134c$. In this case, where the amount of energy for an end mass M_e is given, the $c_p =$ const system is inferior. For the $c_p \sim$ variable system the conditions $g_{\max} = \dfrac{9}{1000}$ and $E/M_e = 8.1 \times 10^{18} =$ "$g_{\max}$"$\cdot c^2$ are identical for large M_a/M_e, of course for a low mass ratio the former condition is more restrictive.

It should, however, be emphasized that at given $g_{\max} = \dfrac{9}{1000}$—when the mission and engine time is not limited—the $c_p =$ constant system delivers an essentially higher u_e and does not imply any restriction the available energy or end velocity except that of quiring that the initial-to-end mass ratio be feasib We can thus compensate to a certain degree for t relatively low exhaust velocity given by $g = \dfrac{9}{1000}$ using a high mass ratio. Of course, however, we cann hope to attain relativistic velocities resulting in essential time dilatation having such low mass co version factor of the fuel.

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Attitude Drift of Space Vehicles

William T. Thomson* and Gordon S. Reiter†

Abstract

The attitude of a body of revolution spinning in the absence of external forces is not a constant when energy dissipation takes place. Elastic vibration, induced by gyroscopic action, results in a dissipation of energy and a change in the precession cone angle θ. This paper examines the effect of energy dissipation on the spinning body and evaluates the time required for a body of given configuration to undergo a specified change in attitude.

I. Introduction

Several proposals have been studied for changing the attitude of spinning space vehicles by using active internal control torques. [1] Attitude changes can also occur in a completely passive system without external moments, since the attitude of a body of revolution spinning in the absence of external forces is not a constant when energy dissipation takes place. This principle has been used in several devices which stabilize spinning satellites by intentional dissipation of energy. [2], [3]

Elastic vibration, induced by gyroscopic action, also results in a dissipation of energy and a change in the precession cone angle θ. This paper examines the effect of energy dissipation due to elastic vibration on a spinning body, and evaluates the time required for a body of given configuration to undergo a specified change in attitude.

The moment-free motion of an unsymmetric body with principal moments of inertia A, B, C, is an unsteady periodic precession and nutation about the resultant angular momentum vector $\bar{h}$ fixed in space. Steady rotation is possible only about the principal axis of maximum or minimum moment of inertia, the principal axis of intermediate moment of inertia being unstable.

For a body of revolution A, A, C, the moment-free motion is a steady precession of the spin axis at a constant angle θ, about the resultant angular momentum vector $\bar{h}$ fixed in space. Steady rotation is again possible about the axis of maximum or minimum moment of inertia, and the principal axis of intermediate moment of inertia does not exist.

In either case, the axis of maximum or minimum moment of inertia is considered to be stable in that if the spin axis deviates slightly from the resultant angular moment vector, there is no tendency of this deviation to grow. This statement is true only for a perfectly rigid body in the absence of external moment.

In an elastic body, deformation between particles will always take place, resulting in some dissipation of energy. When the dissipation of energy is taken into account, we must revise our statement of stability in that it is possible for a small deviation of the spin axis to grow into a large one and eventually result in a complete change in attitude of the body. For such bodies, only the principal axis of maximum moment of inertia is stable, and the axis of minimum moment of inertia is one of unstable equilibrium.

These facts were actually observed in the Explorer I Satellite, [4] which was spin stabilized about the longitudinal axis of minimum moment of inertia. The flexible antennas of the satellite provided an excellent source for energy dissipation, and in one revolution around its orbit (approximately 90 minutes) the Explorer I was observed to be tumbling at an attitude of $\theta = 60°$ instead of spinning about its longitudinal axis at $\theta = 0$. The remedy for this behavior is obviously to shorten the longitudinal dimensions of the satellite so that the moment of inertia about the longitudinal spin axis is greater than that about the transverse pitch or yaw axis. However, the problem still exists for missiles which are long slender bodies and inherently unstable about the spin axis. Here the important question is how long can the spinning missile coast in a moment-free condition without an appreciable change in its attitude.

II. Energy Considerations of Stability

We will now examine the basis for stability from an energy point of view. For a body of revolution with principal moments of inertia A, A, C, as shown in Figure 1, the moment-free motion is that of steady precession described by the equations

$$\dot{\psi} = \frac{C\dot{\phi}}{(A - C)\cos\theta} = \frac{C}{A}\,\frac{\omega_3}{\cos\theta} \qquad (1)$$

$$\omega_3 = \dot{\phi} + \dot{\psi}\cos\theta$$

Since the moment is zero the angular momentum vector $\bar{h}$ is a constant with the following components along the body axes

$$\bar{h} = A(\omega_1\bar{\imath} + \omega_2\bar{\jmath}) + C\omega_3\bar{k} \qquad (2)$$

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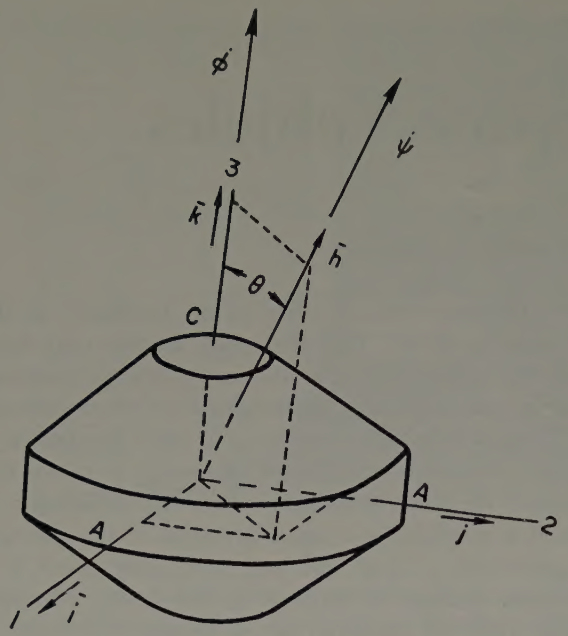

FIG. 1. Coordinate System of Body Axes 1, 2, 3

The magnitude of $\bar h$ can be specified in terms of its initial attitude $\theta \cong 0$, and assuming $\dot\theta$ to be small, the following relationships must hold.

$$
\begin{aligned}
h &= C\omega_0 \\
\omega_3 &= \omega_0 \cos\theta \\
\dot\psi &= \frac{C}{A}\,\omega_0 \\
\dot\phi &= \left(1 - \frac{C}{A}\right)\omega_0 \cos\theta
\end{aligned}
\qquad (3)
$$

We must examine the kinetic energy of rotation

$$T = \tfrac{1}{2}A(\omega_1^2 + \omega_2^2) + \tfrac{1}{2}C\omega_3^2$$

For $\dot\theta$ small, we have

$$
\begin{aligned}
A\sqrt{\omega_1^2 + \omega_2^2} &= h \sin\theta \\
C\omega_3 &= h\cos\theta
\end{aligned}
\qquad (4)
$$

so that T can be expressed in the form

$$T = \frac{1}{2}\,C\omega_0^2\left[1 + \left(\frac{C}{A} - 1\right)\sin^2\theta\right] \qquad (5)$$

Equation (5) indicates that θ remains constant provided T is constant. However with dissipation of energy T must decrease. Differentiating (5) we have, as in References [5] and [6]

$$\dot T = C\omega_0^2\left(\frac{C}{A} - 1\right)\sin\theta\cos\theta\,\dot\theta \qquad (6)$$

and since $\dot T$ must always be negative, $\dot\theta$ is negative for $(C/A) > 1$ and positive for $(C/A) < 1$. Thus the principal axis of minimum moment of inertia is one of unstable equilibrium, and a small deviation of th spin axis will increase due to energy dissipation wh $(C/A) < 1$.

III. Dissipation of Energy

Assuming an elastic body, the energy dissipated p unit volume per cycle of stress can be assumed to be

$$\frac{\gamma\sigma^2}{2E} \qquad (7$$

where γ is a hysteretic damping factor establishing th fraction of the elastic energy which is dissipated shown by the shaded area in Fig. 2. Dividing by th time t_0 per cycle of stress, and integrating over th entire structure, the rate of energy dissipation can b found. Thus the equation to be solved is of the gener form

$$-\int \frac{\gamma\sigma^2}{2Et_0}\,dV = C\omega_0^2\left(\frac{C}{A} - 1\right)\sin\theta\cos\theta\,\dot\theta \qquad (8$$

In examining the source of cyclic stressing, fre vibration can be discarded since it will soon damp ou Steady cycling of stress is however induced by th gyroscopic precession, and these stresses are repeate at the rate $\dot\phi$ and $2\dot\phi$, as we will presently show.

The excitation for the cyclic stress is the accelera tion. Choosing an arbitrary point on the structure an orienting the plane 1, 0, 3, through it, the displacemen vector for the specified point is

$$\bar r = \xi\bar\imath + z\bar k \qquad (9$$

With $\dot\theta$ small, the angular velocity and acceleration the coordinate axes 1, 2, 3 are

$$\bar\omega = \dot\psi\sin\theta\sin\phi\,\bar\imath + \dot\psi\sin\theta\cos\phi\,\bar j + (\dot\phi + \dot\psi\cos\theta)\bar k \quad (10$$

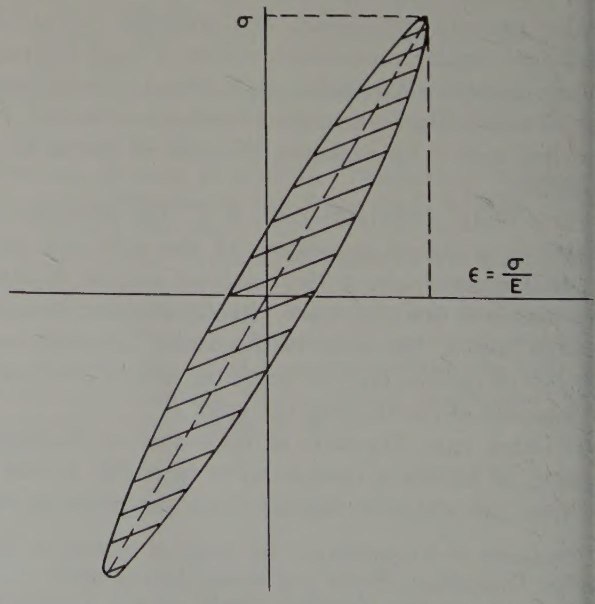

FIG. 2. Energy Dissipated by Hysteretic Damping

$$\dot{\bar{\omega}} = \left(\frac{d\bar{\omega}}{dt}\right)_{i,j,k,=\text{const}} + \bar{\omega} \times \bar{\omega} \qquad (11)$$
$$= \dot{\phi}\dot{\psi} \sin \theta \left(\cos \phi \bar{\imath} - \sin \phi \bar{\jmath}\right)$$

Substituting into the general vector equation for the acceleration

$$\bar{a} = \bar{a}_0 + \bar{a}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}) + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times \bar{v}' \qquad (12)$$

and noting that the following quantities are zero

$$\bar{a}_0 = \bar{a}' = \bar{v}' = 0$$

the result after some algebraic reduction is

$$\begin{aligned} \bar{a} = {} & [-\xi(\dot{\phi}^2 + \dot{\psi}^2) + \xi\dot{\psi}^2 \sin^2 \theta \sin^2 \phi \\ & - 2\xi\dot{\phi}\dot{\psi} \cos \theta + z\dot{\psi}^2 \sin \theta \cos \theta \sin \phi]\bar{\imath} \\ & + (\xi\dot{\psi}^2 \sin^2 \theta \sin \phi \cos \phi \\ & \qquad + z\dot{\psi}^2 \sin \theta \cos \theta \cos \phi)\bar{\jmath} \\ & + (2\xi\dot{\phi}\dot{\psi} \sin \theta \sin \phi \\ & \qquad + \xi\dot{\psi}^2 \sin \theta \cos \theta \sin \phi - z\dot{\psi}^2 \sin^2 \theta)\bar{k} \end{aligned} \qquad (13)$$

A somewhat more convenient form of (13) results by eliminating $\dot{\phi}$ and $\dot{\psi}$ by means of (3).

$$\begin{aligned} \bar{a} = {} & \omega_0^2 \left[- \xi \left(\frac{C}{A}\right)^2 + \xi \left(\frac{C}{A}\right)^2 \sin^2\theta \sin^2\phi \right. \\ & \left. + \xi\left\{\left(\frac{C}{A}\right)^2 - 1 \right\}\cos^2 \theta + z\left(\frac{C}{A}\right)^2 \sin\theta\cos\theta\sin\phi \right]\bar{\imath} \\ & + \omega_0^2 \left[\xi\left(\frac{C}{A}\right)^2 \sin^2 \theta \sin \phi \cos \phi + z\left(\frac{C}{A}\right)^2 \sin\theta \right. \\ & \left. \cdot \cos\theta\cos\phi \right]\bar{\jmath} + \omega_0^2\left[- \xi\left(\frac{C}{A}\right)\left(\frac{C}{A} - 2\right)\sin\theta \right. \\ & \left. \cdot \cos\theta\sin\phi - z\left(\frac{C}{A}\right)^2 \sin^2 \theta \right] \bar{k} \end{aligned} \qquad (14)$$

Since the only time varying quantity in (14) (assuming $\dot{\theta}$ to be negligible) is $\phi = \dot{\phi}t$, it is evident that the cyclic stress is repeated at a rate $\dot{\phi}$ and $2\dot{\phi}$. It should be pointed out that for slender bodies like missiles, C/A is small compared to unity and the predominant variable acceleration term is

$$\bar{a}_w = 2\omega_0^2\, \xi \left(\frac{C}{A}\right) \sin \theta \cos \theta \sin \phi \cdot \bar{k} \qquad (15)$$

which is repeated in the time

$$\begin{aligned} t_0 = \frac{2\pi}{\dot{\phi}} &= \frac{2\pi}{\left(1 - \dfrac{C}{A}\right)\omega_0 \cos\theta} \quad \text{when} \quad C < A \\ &= \frac{2\pi}{\left(\dfrac{C}{A} - 1\right)\omega_0 \cos\theta} \quad \text{when} \quad C > A \end{aligned} \qquad (16)$$

IV. Example

As an example of the simplest kind, we will consider two solid disks connected by a flexible tube as shown

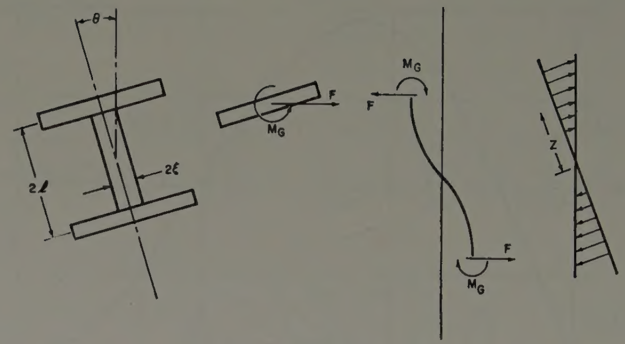

Fig. 3. Satellite Configuration, Displacement and Moment Distribution.

in Fig. 3. We will let C_1 and A_1 be the moment of inertia of each disk about its own polar and diametric axes. The gyroscopic moment required by each disk is

$$M_G = C_1(\dot{\phi} + \dot{\psi} \cos \theta)\dot{\psi} \sin \theta - A_1\dot{\psi}^2 \sin \theta \cos \theta \quad (17)$$

Since the moments of inertia about the center of mass of the body are

$$C = 2C_1$$
$$A \cong 2(A_1 + m_1l^2)$$

Equation (17) can be rewritten as

$$M_G = \tfrac{1}{2}[C(\dot{\phi} + \dot{\psi} \cos \theta)\dot{\psi} \sin \theta - A\dot{\psi} \sin \theta \cos \theta] + m_1l^2\dot{\psi}^2 \sin \theta \cos \theta$$

The first term, however, is the moment about the center of mass which is zero and from which (1) is obtained. We are thus left with

$$\begin{aligned} M_G &= m_1l^2\dot{\psi}^2 \sin \theta \cos \theta \\ &= Fl \cos \theta \end{aligned} \qquad (18)$$

where $F = m_1l\dot{\psi}^2 \sin \theta$ is the centripetal force of the precessing disk.

The effect of M_G and F on the flexible tube is shown in Fig. 3. At point z along the tube, measured from the center of mass, the bending moment is

$$M_z = M_G \frac{z}{l} \qquad (19)$$

and the expression for the maximum stress becomes

$$\begin{aligned} \sigma = \frac{M_z\, \xi}{I} &= m_1\, l^2\dot{\psi}^2 \frac{z}{l}\frac{\xi}{I} \sin \theta \cos \theta \\ &= \frac{1}{2}\, ml^2 \left(\frac{C}{A}\right)^2 \omega_0^{\,2} \frac{z}{l}\frac{\xi}{I} \sin \theta \cos \theta \end{aligned} \qquad (20)$$

which is repeated at the rate given by (16). The rate of energy dissipation as given by the left side of (8) is then

$$\frac{\gamma}{48\pi E}\left(\frac{ml^2\xi}{I}\right)^2 V \left(\frac{C}{A}\right)^4\left(\frac{C}{A} - 1\right) \omega_0^{\,5} \sin^2\theta \cos^3\theta \quad (21)$$

and the rate of change of the attitude angle θ becomes

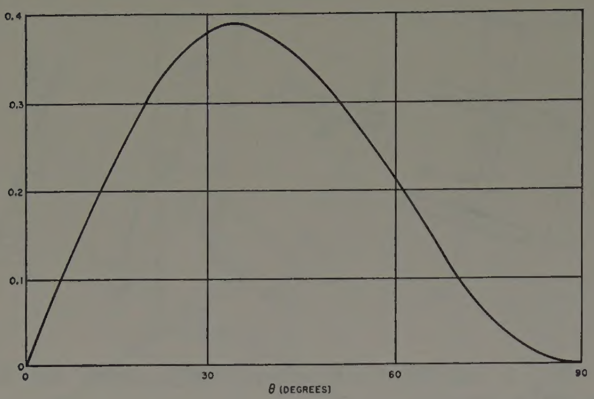

FIG. 4. Variation in the Rate of Tumbling

$$\dot{\theta} = \frac{\gamma}{48\pi E}\left(\frac{ml^2\xi}{I}\right)^2 \frac{V}{C}\left(\frac{C}{A}\right)^4 \omega_0^{\ 3} \sin\theta\cos^2\theta$$
$$= K\sin\theta\cos^2\theta \qquad (22)$$

A plot of (22) is shown in Fig. 4. Since $\dot{\theta}$ is zero for $\theta = 0$, tumbling cannot be initiated unless the initial value of θ is finite. However $\theta = 0$ is never attainable in practice for many reasons, and $\dot{\theta}$ will build up when C/A is less than unity. By differentiating (22), $\dot{\theta}$ can be shown to have a maximum at $\theta = \tan^{-1}(1/\sqrt{2}) = 35° 20'$. Due to $\cos^2\theta$, $\dot{\theta}$ will diminish to a small value near $\theta = 90°$, and an infinite time will be required to reach this angle.

For small values of θ, (22) is approximately equal to

$$\dot{\theta} = K\theta \qquad (23)$$

and the time required for the attitude angle to change from θ_0 to θ_1 is

$$t = \frac{1}{K}\ln\frac{\theta_1}{\theta_0} \qquad (24)$$

Numerical Example

Let the two solid disks be aluminum, $\frac{1}{2}$ inch thick and 24 inches diameter, and the flexible tube be 0.032 inch stainless steel 6 inches in diameter and 24 inches long. The quantities required for the computation of K are:

$$\begin{aligned} C &= 8.16 \text{ lb in. sec}^2 \\ A &= 20.44 \text{ lb in. sec}^2 \\ m &= 0.1136 \text{ lb in.}^{-1}\text{ sec}^2 \\ V &= 14.5 \text{ in.}^3 \\ \xi &= 3.0 \text{ in.} \\ I &= 2.71 \text{ in.}^4 \\ l &= 12.0 \text{ in.} \\ E &= 29 \times 10^6 \text{ lb in.}^{-2} \end{aligned}$$

Assuming $\gamma = 0.05$ and $\omega_0 = 50\pi$, the value of K is 662×10^6. Thus for the body to undergo an attitude change from 1° to 10°, the time required, as calculated from (24) is

$$t = \frac{2.303}{662} \times 10^6 = 3480 \text{ sec}$$
$$= \underline{\underline{58.0 \text{ min}}}$$

V. Example Including Inertia Forces

In the preceding section, inertia forces due to th *elastic* motion were neglected, and so no resonanc phenomenon is observed in the solution. A problen where inertia effects are important is the followin (Fig. 5).

The rotating cylinder of radius R is rigid. Four un form dissipative, elastic cantilever beams project fro its center at right angles to the axis of symmetry. Th beams bend only in the z direction. Assuming the elasti deformations $w(x, t)$ to be small compared to the rigi body motion, the equation of motion of each beam is

$$EI\frac{\partial^4 w}{\partial x^4} + \frac{M}{L}\frac{\partial^2 w}{\partial t^2} = a_w\frac{M}{L} \qquad (25$$

where a_w is given by (15) and ξ has been replaced b $x + R$ (Fig. 5). EI is the flexural rigidity of one of th beams, M is the beam mass, L is the beam length.

The deflection $w(x, t)$ may be expressed as a serie of products of generalized coordinates $q_n(t)$ and th normal functions $\psi_n(x)$ of the beam,

$$w(x, t) = \sum_{n=1}^{\infty} q_n(t)\psi_n(x) \qquad (26$$

The normal functions $\psi_n(x)$ are assumed normalize to the beam length, i.e.

$$\int_0^L \psi_n^{\ 2}\, dx = L \qquad (27$$

Substituting (26) into (28) and using the fact that th normal functions are orthogonal the equations

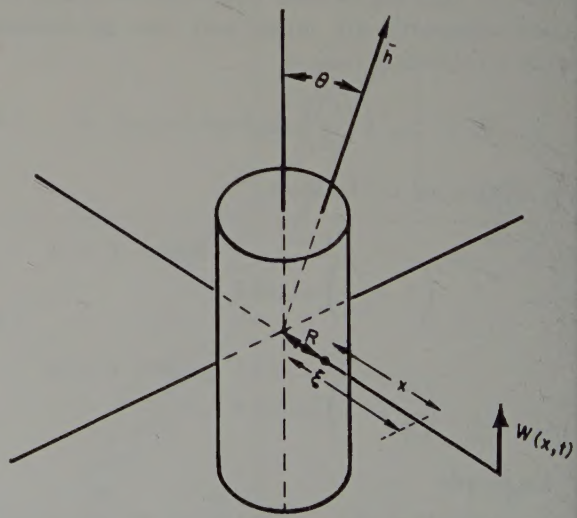

FIG. 5. Spinning Cylinder with Projecting Beams

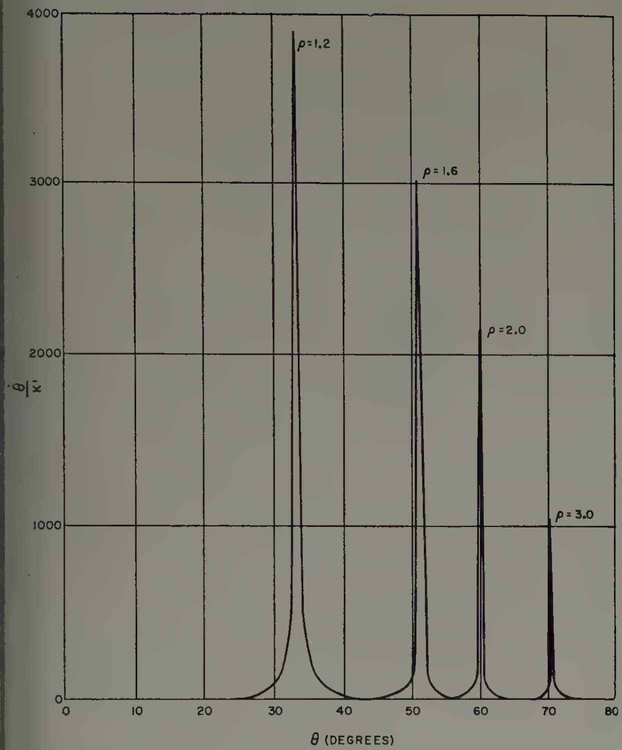

FIG. 6. Variation in the Rate of Tumbling including Resonance Effects.

motion become

$$\ddot{q}_n(t) + \left(1 + i\,\frac{\gamma}{2\pi}\right)\Omega_n^{\,2}\, q_n(t) = \frac{1}{L}\int_0^L \psi_n(x)\, a_w\, dx \qquad (28)$$

where a complex structural damping factor $i\gamma/2\pi$ has been assumed, and Ω_n is the nth natural frequency. After substitution of a_w from (15), the integral in (28) can be evaluated from the tables in [7] using the constants α_n and β_n tabulated in [8], to give

$$\ddot{q}_n(t) + \left[1 + i\left(\frac{\gamma}{2\pi}\right)\right]\Omega_n^{\,2}\, q_n(t) = \frac{4\omega_0^{\,2} C}{AL\beta_n^{\,2}}\,(1 + R\alpha_n\,\beta_n)\sin\dot{\phi}t\,\sin\theta\cos\theta \qquad (29)$$

Solving this forced vibration equation and integrating the damping force for each mode over one precession cycle, the over-all dissipation per cycle is given by

$$\Delta T = \frac{32C^2 M\gamma\omega_0^{\,4}\sin^2\theta\cos^2\theta}{A^2 L^2}\sum_{n=1}^{\infty} \cdot \frac{(\alpha_n\,\beta_n\, R + 1)^2}{\left\{\left[1 - \left(\frac{\dot{\phi}}{\Omega_n}\right)^2\right]^2 + \left(\frac{\gamma}{2\pi}\right)^2\right\}\beta_n^{\,4}\,\Omega_n^{\,2}} \qquad (30)$$

where the dissipation has been increased by a factor of four because there are four beams. Since the quantity $\beta_n^{\,4}$ increases very rapidly with increasing n, all but the first term of the above series will be neglected. Then using (3), (8), and (16) the equation for the cone angle drift rate is

$$\dot{\theta} = \frac{16CM\gamma(\alpha_1\,\beta_1\, R + 1)^2\,\omega_0^{\,3}\sin\theta\cos^2\theta}{\pi A^2\beta_1^{\,4} L^2\Omega_1^{\,2}\left(\left\{1 - \left[\left(1 - \frac{C}{A}\right)\frac{\omega_0}{\Omega_1}\cos\theta\right]^2\right\}^2 + \left(\frac{\gamma}{2\pi}\right)^2\right)} \qquad (31)$$

or

$$\dot{\theta} = \frac{K'\sin\theta\cos^2\theta}{\left[(1 - \rho^2\cos^2\theta)^2 + \left(\frac{\gamma}{2\pi}\right)^2\right]} \qquad (32)$$

where

$$\rho = \left(1 - \frac{C}{A}\right)\frac{\omega_0}{\Omega_1} \qquad (33)$$

Note that the resonance term in the denominator of (32) allows $\dot{\theta}$ to become large if the forcing frequency $\dot{\phi}$ coincides with the beam natural frequency Ω_1 for some θ. This effect will be present for all $\rho > 1$. The effect is illustrated in Fig. 6, which is a plot of (32) for the case $(\gamma/2\pi) = 0.01$, $\rho = 1.2$, 1.6, and 2. Note that the envelope of the peaks in Fig. 6 is identical in shape to Fig. 4.

The solution to (32) is

$$K't = \left[1 + \left(\frac{\gamma}{2\pi}\right)^2\right](\sec\theta - \sec\theta_0) + \left[1 + \left(\frac{\gamma}{2\pi}\right)^2 + \rho^4 - 2\rho^2\right]\ln\left|\frac{\tan\frac{\theta}{2}}{\tan\frac{\theta_0}{2}}\right| + \rho^4(\cos\theta - \cos\theta_0) \qquad (34)$$

Equation (34) is plotted in Fig. 7 for the case $(\gamma/2\pi) = 0.01$, $\rho = 1.2$, 1.6 and 2.0, $\theta(0) =$ one degree. The resonant behavior in these cases causes the angle to diverge quite rapidly. For some cases involving lightly damped resonance the assumption that θ is slowly varying, made in this analysis, may not hold.

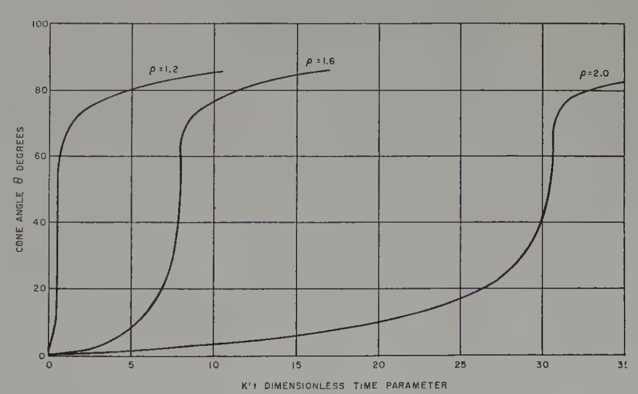

FIG. 7. Attitude Change of Unstable Body including Resonance Effects.

VI. Conclusions

1. The attitude change of an unstable, dissipative body can be predicted by calculating the oscillating stress field in the body, and relating it to the energy dissipated.

2. The acceleration given by (14), used as inertia force per unit mass, and used in the appropriate differential equation for the structure, will result in the amplitude of cyclic vibration from which the rate of energy dissipation can be determined.

3. If only the z component of acceleration is involved, as in the illustrative examples, the equation for θ will have the form of (32) where K' will take on different numerical values depending on the configuration. If inertia forces due to the elastic motion are neglected, the equation for θ will have the form of (22).

4. The assumption made in the derivation of equations other than (23) and (24) is that $\dot{\theta}$ is small. This assumption is justifiable in most cases, but may be violated, for some cases, if resonance is important.

VII. Nomenclature

A, C = moment of inertia about pitch and spin axes respectively
$\bar{h}$ = angular momentum vector
θ = spin axis attitude measured from $\bar{h}$
$\dot{\phi}$ = spin rate
$\dot{\psi}$ = precession rate
1, 2, 3 = body axes
$\omega_{i,j,k,}$ = angular velocity about body axes (1, 2, 3)
ω_0 = initial angular velocity for $\theta = 0$
T = kinetic energy
σ = maximum cyclic stress
E = modulus of elasticity
γ = hysteretic damping factor = $2\pi \times$ solid damping coefficient
t_0 = cyclic time
ξ = coordinate along axis 1
z = coordinate along axis 3
V = volume of stressed material
$\bar{a}$ = acceleration vector
$\bar{a}_0$ = acceleration of origin of axes 1, 2, 3
$\bar{a}'$ = acceleration relative to axes 1, 2, 3
$\bar{v}'$ = velocity relative to axes 1, 2, 3
$\bar{\omega}, \dot{\bar{\omega}}$ = angular velocity and acceleration of coordinate axes 1, 2, 3
M_G = gyroscopic momnet on one disk
m_1 = mass of one disk
m = $2m_1$ = total mass (of 2 disks)
l = length of flexed tube
K = constant defined by Equation (22)
R = radius of spinning cylinder
I = beam cross-section moment of inertia
M = beam mass
L = beam length
$w(x, t)$ = beam deflection
Ω_n = nth natural frequency of beam
$\psi_n(x)$ = nth normal function of beam, correspondin to
$q_n(t)$ = nth generalized coordinate of beam deflection
i = $\sqrt{-1}$
α_n, β_n = constants for uniform beam, tabulated i Reference [6]
K' = constant defined by Equations (31) ar (32)
ρ = frequency ratio defined by Equation (33

Dots denote derivatives with respect to time.

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Normal Lunar Impact Analysis

Anthony Liu*

Abstract

If the angle of impact is chosen to be perpendicular to the local lunar surface, (i.e. normal impact), for purposes of simplifying terminal guidance of lunar vehicles, then it is possible to show, on the basis of a rocket moving under the gravitational field of the Earth and a circularly moving but massless Moon, that normal impact region is restricted to the leading semi-circle of the visible disc. Further, it is possible to show that higher energy orbits will strike closer to the center of the visible lunar disc than lower energy orbits.

Discussion

A lunar orbit is characterized by the flight time required for the rocket to impact the surface of the Moon. Clearly, a flight time of $1\frac{1}{4}$ days implies a higher orbital energy than a flight time of $3\frac{1}{4}$ days or 4 days. If the condition of normal impact, that the angle of impact of the vehicle be perpendicular to the lunar surface, be invoked, then, it will be shown in this paper, higher energy orbits strike closer to the center of the lunar disc than lower energy orbits. Further we designate areas on the Moon to be reached by elliptical, hyperbolic, or parabolic orbits for normal impacts depending upon the initial conditions near the Earth.

To show this, we assume:

(1) Normal impact on the lunar surface;

(2) Spherical non-accelerating Earth;

(3) Massless Moon moving in a circular geocentric orbit;

(4) Two-body geocentric motion of rocket;

(5) Orbit plane of the Moon coincident with the equatorial plane of the Moon;

(6) A translating but non-rotating Moon;

(7) At final impact conditions, the distance from rocket to Earth is the same as Moon to the Earth; and

(8) Rocket burnout at "reasonable" altitudes and speed.

By considering both the geometry and dynamics of the Earth-Moon-rocket configuration, together with the above-listed assumptions, equations involving the lunar impact longitude (for the coplanar case) and longitude and latitude (for the three-dimensional) can be derived in terms of gravitational constraints. Certain regions on the lunar surface can be designated as "elliptical regions," others as "hyperbolic regions" and the boundary between the two regions designated as "parabolic region." In order to find limits for the normal impact latitude, it is necessary to find the maximum value of p, the semi-latus rectum, by use of physical constraints. The results of the analysis will

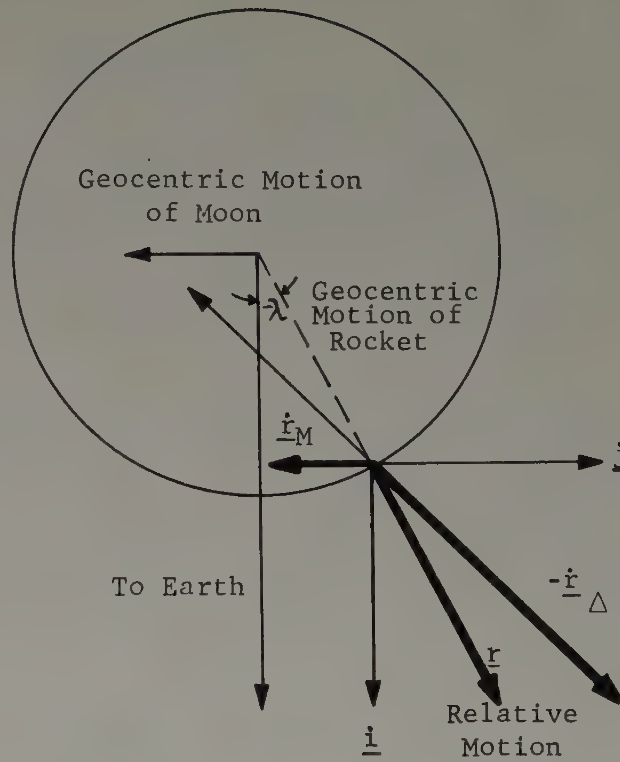

FIG. 1. Relative velocity of moon and rocket

show that the effect of p upon longitude, serves to "fuzz out" the sharp separation of the areas found to exist for rectilinear cases, but the general characteristics still hold; i.e., higher energy orbits impact closer to the center of the lunar disc than lower energy orbits.

This analysis, taken together with the study of the effect of retrograde and direct orbits upon latitude will also show that for retrograde orbits the hyperbolic region assumes one shape (the remaining area occupied by the elliptical region); while for direct orbits the hyperbolic region assumes another shape (with the remaining area occupied by the elliptical region).

First, consider a simple model of massless rocket impacting normally at a longitude λ, upon a massless Moon moving on a circular orbit. Both are traveling under the gravitational influence of a spherical, non-accelerated Earth. The justification for the neglect of the Moon's gravitational influence upon the rocket is that the effect of relative geocentric velocity between Moon and rocket is the principal effect, while lunar perturbations are of second order. Lunar rotation is also neglected so that the angle of impact is calculated in a non-rotating coordinate frame.

From Fig. 1 we see the different roles played by the velocity of the Moon and of the incoming rocket. Let

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the following quantities be defined:

$\dot{\underline{r}}_\Delta$ = geocentric velocity of the rocket
$\dot{\underline{r}}_M$ = geocentric velocity of the Moon
$\dot{\underline{r}}$ = relative velocity of Moon with respect to rocket
λ = longitude of the impact point measured positively eastward from the Earth-Moon line
ϕ = latitude of the impact point measured positively northward from the Moon's equator
r_Δ = distance of rocket to Earth
v = true anomaly measured from perigee positively counterclockwise to the object
$\dot{s}_\Delta$ = geocentric speed of rocket
$\dot{s}_M$ = geocentric speed of Moon
$\dot{s}$ = relative speed of Moon with respect to rocket
p = semi-latus rectum of geocentric orbit in Earth radii
a = semi-major axis of geocentric orbit in Earth radii

The units adopted here are "normalized" units; i.e., the unit of time is a k_e^{-1} minute (13.447 minutes) and unit mass is the mass of the Earth, and the unit of length is one Earth equatorial radius.

The requirements that impact be normal to the surface of the Moon implies that the velocity vector $\dot{\underline{r}}$ must pass through the center of the Moon. Also, since the Moon's rotation is neglected, we note that every point on the surface of the Moon partakes of the same velocity as the velocity of the center.

Figure 2 illustrates the assumed rocket trajectory.

By treating the Moon as a sphere and not a disc in order to study latitude of impact, the following situation is depicted by Fig. 3 where an additional definition is introduced, namely I, the angle between the geocentric orbital plane and the Moon's equatorial plane.

In Fig. 3, the only vector that lies in the orbital plane of the vehicle is $\dot{\underline{r}}_\Delta$. Since the rocket's orbit plane must contain the center of the Earth, the radial component of $\dot{\underline{r}}_\Delta$ is nearly along $\underline{i}$. The transverse component

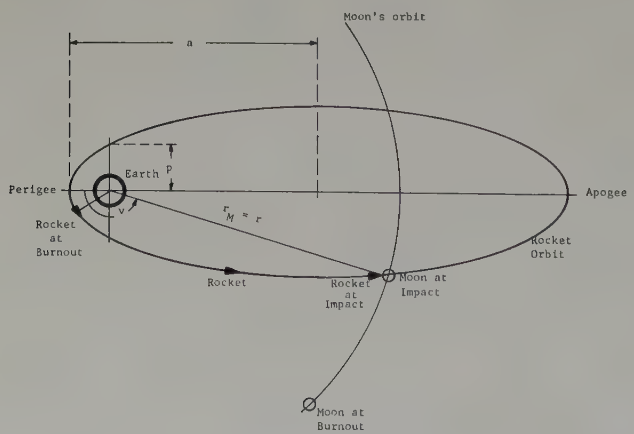

FIG. 2. Assumed trajectory of rocket

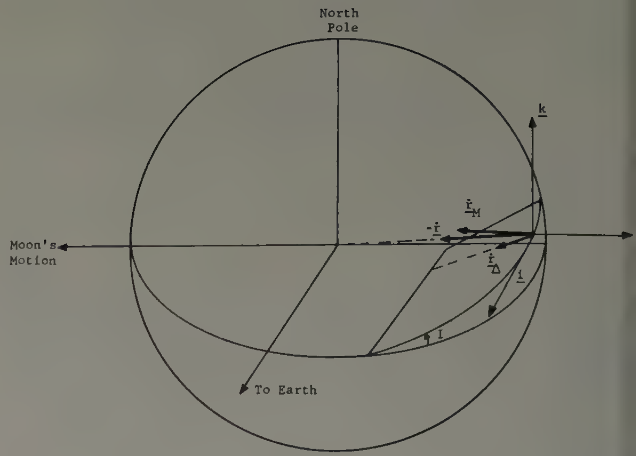

FIG. 3. Angle between geocentric orbital plane and moon equatorial plane.

of $\dot{\underline{r}}_\Delta$ however lies in the vehicle's orbital plane, parall to the $\underline{k}$, $\underline{j}$ plane, and can be resolved into two furthe components, one along $\underline{j}$ and another along $\underline{k}$. Th normalcy condition requires that $\dot{\underline{r}}$ pass through th geometrical center of the Moon. The Moon's motion i a circular orbit about the Earth is along the negativ $\underline{j}$ axis.

Three vector equations represent the velocity of th Moon with respect to the vehicle, and of the Moon an vehicle with respect to the Earth as follows:

$$\dot{\underline{r}} = \underline{i}\dot{s}\cos\phi\cos\lambda + \underline{j}\dot{s}\cos\phi\sin\lambda + \underline{k}\dot{s}\sin\phi \quad \text{(1a}$$

$$\dot{\underline{r}}_M = -\underline{j}\dot{s}_M \quad \text{(1b}$$

$$\dot{\underline{r}}_\Delta = -\underline{i}\dot{r}_\Delta - \underline{j}(r\dot{v})_\Delta\cos I - \underline{k}(r\dot{v})_\Delta\sin I \quad \text{(1c}$$

For the requirement that $\dot{\underline{r}} = \dot{\underline{r}}_M - \dot{\underline{r}}_\Delta$, the correspond ing scalar equations representing the motion of th Moon with relation to the rocket will then follow:

$$\dot{s}\cos\phi\cos\lambda = \dot{r}_\Delta \quad \text{(2a}$$

$$\dot{s}\cos\phi\sin\lambda = (r\dot{v})_\Delta\cos I - \dot{s}_M \quad \text{(2b}$$

$$\dot{s}\sin\phi = +(r\dot{v})_\Delta\sin I \quad \text{(2c}$$

In terms of the geocentric orbital constants, the ex pressions for ϕ and λ are as follows:

$$\tan\lambda = \left[\pm\left(\sqrt{\frac{p}{r}}\cos I\right) - 1\right] \Big/ \left[2 - \frac{r}{a} - \frac{p}{r}\right]^{1/2} \quad \text{(3a}$$

$$\tan\phi = \pm\left(\sqrt{\frac{p}{r}}\sin I\right) \Big/ \left[2 - \frac{r}{a} - \frac{p}{r} + \left(\sqrt{\frac{p}{r}}\cos I - 1\right)^2\right]^{1/2} \quad \text{(3b}$$

An upper limit of p, the semi-latus rectum, exist from purely practical restrictions. The value of p determined *uniquely* from *initial* burnout condition In fact, in the normalized units, p is equal to the squar of the angular momentum. The amount of angul

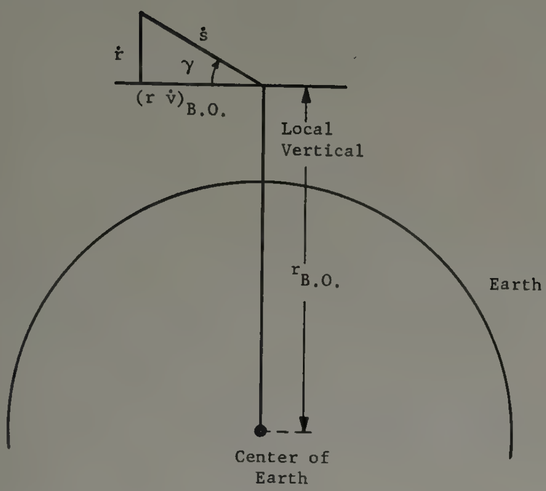

Fig. 4. Projected angle of rocket from local horizon

momentum imparted to a vehicle initially is limited so that p is certainly limited. This may best be seen by the following considerations. From Fig. 4 it can be seen that:

γ = initial angle of projection or path angle of the rocket from the local horizon
$\dot{v}$ = time rate of change of the *true anomaly*
$r_{B.O.}$ = distance of rocket at burnout from the center of the Earth.

$$(r\dot{v})_{B.O.} = \dot{s}_{B.O.} \cos \gamma.$$

But by conservation of angular momentum we have:

$$(r^4\dot{v}^2)_{B.O.} = p = \text{constant}$$
$$p = r^2_{B.O.}\dot{s}^2_{B.O.} \cos^2 \gamma$$
$$p_{\max} = (r_{B.O.})^2_{\max}(\dot{s}^2_{B.O.})_{\max}$$

For physically realizable missile systems, we might put as upper bounds for $r_{B.O.}$ and $\dot{s}$ the following:

$(r_{B.O.})_{\max}$ = 1.3 Earth radii (i.e., 1200 miles above surface of Earth)
$r_{\max}$ = 1.5 radii/k_e^{-1} min (i.e., 39,000 ft/sec)
$p_{\max}$ = $(1.3)^2\ (1.5)^2 < 4$ Earth radii.

It is noted that present-day rockets do not burn out at altitudes of 1,200 miles with speeds in excess of 39,000 feet/sec. Thus, for reasonable lunar trajectories, p is less than 4 Earth radii. P/r is then less than 0.1. Since

$$s_M = \sqrt{\frac{1}{r}}$$

and,

$$(r\dot{v})_\Delta = \sqrt{\frac{p_{\max}}{r}} \cdot \sqrt{\frac{1}{r}}$$
$$(r\dot{v})_\Delta < s_M \sqrt{.1}$$

It can be seen that s_M is always greater than $(r\dot{v})_\Delta \cos I$. This fact is useful in removing the sign ambiguity in equations 3a and 3b by the following argument.

Let the angle of inclination, I, of the relative orbit planes always be measured at the ascending node determined by a direct orbit (inclination is less than 90°). All inclination angles hereinafter are always measured at *this* node so that I may take on values from 0° to 360°. (The normal convention in astronomy is to measure I *always* at the *ascending node*.) Consider that in Eq 2b

(A) $\dot{s}$ is always positive;
(B) Since $-90 < \phi < +90$, $\cos \phi$ is always positive;
(C) $\dot{s}_M > (r\dot{v})_\Delta \cos I$.

Thus, the right-hand side of Eq. 2b is always negative. Neglecting the case of intersection of the Moon's orbit after apogee, so that $\dot{r}_\Delta$ is positive, we can conclude that λ is restricted to values between $-90°$ and 0°. In Eq. 3a, the plus sign is valid.

If Eq. 2b is divided by Eq. 2c, one obtains

$$\cot \phi = \frac{(r\dot{v})_\Delta \cos I - \dot{s}_M}{(r\dot{v})_\Delta \sin I \sin \lambda}$$

The numerator is always negative, as shown above. Therefore,

$$\cot \phi = \frac{(\textit{negative quantity})}{\sin I\ (\textit{positive quantity})\ (\textit{negative quantity})}$$
$$= \frac{(\textit{positive quantity})}{\sin I}$$

The extreme value of $p = 4$ determined previously is one consideration needed to calculate the maximum value of ϕ obtainable by a ballistic missile. Other values of ϕ and λ, for combinations of p, a, and I, are given in Table I, and Fig. 5A and 5B show graphically the possible variation of ϕ and λ with a.

Note that in Fig. 5A, since physically realizable rocket configuration limit p to 4 radii, under the most favorable circumstances the maximum absolute latitude obtainable is on the order of ±15° for normal impact. In Fig. 5B, the upper branches of the family of curves do not appear because the angular momentum of the Moon is always greater than the angular momentum of the rocket. Also note that by various choices of p between 0 and 4, and I, a certain amount of overlap of normal impact areas is possible.

Figures 6, 7, 8 and 9, obtained by plotting the values given in Table I, show the areas where normal impacts may occur for various trajectory characteristics and also show the locus of all normal impact points for the parabola. The area is over a restricted portion of the Moon's surface, bounded by longitudes of −90° to 0° and latitudes of about ±15°. The areas for longitudes from −90° to −35.3° can be considered effectively as elliptical (with the exception of the inclination effect which serves to shift the areas to the right of −35.3° for inclinations between 270° and 360° and 0° to 90° and to the left of −35.3° for inclinations between 90°

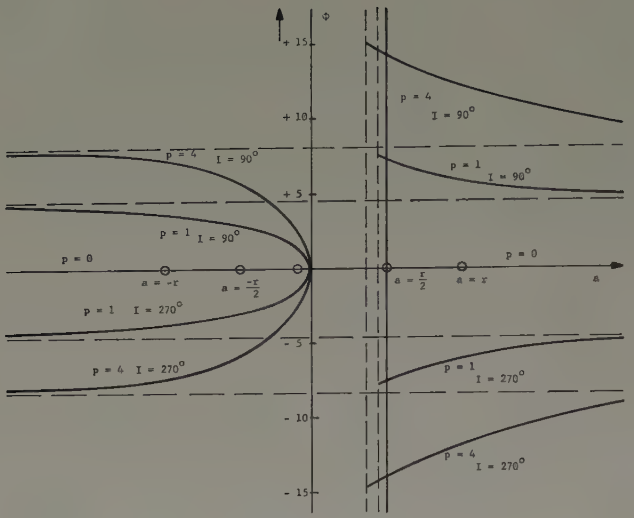

FIG. 5A. Latitude vs. semi-major axis

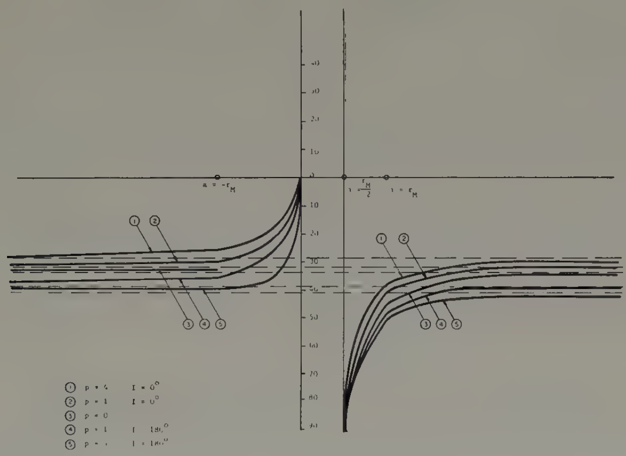

FIG. 5B. Plot of longitude vs. semi-major axis

and 270°), and the areas from $-35.3°$ to 0° as effectively hyperbolic. Also, as can be seen from Fig. 6 through 9, for what is usually termed as direct orbits; i.e., $0° < I < 90°$; $270 < I < 360°$, the hyperbolic regions take one shape while for what is conventionally termed as retrograde orbit; i.e., $90° < I < 270°$, the hyperbolic region assumes another shape. $p = 0$ is not shown, since on these figures, the plot will merely be a straight line falling upon the equator.

TABLE I

p = 0 (Rectilinear Orbit)

a	r_M	$+r_M/2$	$\pm\infty$	$-r_M$	0
λ	$-45°\!.0$	$-90°\!.0$	$-35°\!.3$	$-30°\!.0$	0°

p = 1

a	r_M		$\dfrac{r_M}{2-\dfrac{1}{r_M}}$		$\pm\infty$		r_M		0	
I	λ	φ	λ	φ	λ	φ	λ	φ	λ	φ
0°	$-41°\!.2$	0°	$-90°$	0°	$-32°\!.8$	0°	$-26°\!.8$	0°	0°	0°
90°	$-45°\!.0$	$+5°\!.1$	$-90°$	$+7°\!.2$	$-35°\!.3$	$+4°\!.2$	$-30°\!.0$	$+3°\!.6$	0°	0°
180°	$-48°\!.4$	0°	$-90°$	0°	$-38°\!.5$	0°	$-33°\!.0$	0°	0°	0°
270°	$-45°\!.0$	$-5°\!.1$	$-90°$	$-7°\!.2$	$-35°\!.3$	$-4°\!.2$	$-30°\!.0$	$-3°\!.6$	0°	0°

p = 4

a	r_M		$\dfrac{r_M}{2-\dfrac{4}{r_M}}$		$\pm\infty$		$-r_M$		0	
I	λ	φ	λ	φ	λ	φ	λ	φ	λ	φ
0°	$-36°\!.0$	0°	$-90°$	0°	$-27°\!.9$	0°	$-23°\!.4$	0°	0°	0°
90°	$-45°\!.0$	$+10°\!.2$	$-90°$	$+14°\!.2$	$-35°\!.3$	$+8°\!.2$	$-30°\!.0$	$+7°\!.2$	0°	0°
180°	$-51°\!.3$	0°	$-90°$	0°	$-41°\!.5$	0°	$-35°\!.8$	0°	0°	0°
270°	$-45°\!.0$	$-10°\!.2$	$-90°$	$-14°\!.2$	$-35°\!.3$	$-8°\!.2$	$-30°\!.0$	$-7°\!.2$	0°	0°

Acknowledgments

The author expresses deep gratitude to Dr. R. M. L. Baker, Jr., who suggested the problem, Mr. G. Matlin, Dr. L. G. Walters and the Astrodynamic Staff of Aeronutronic for much valuable aid and fruitful discussion.

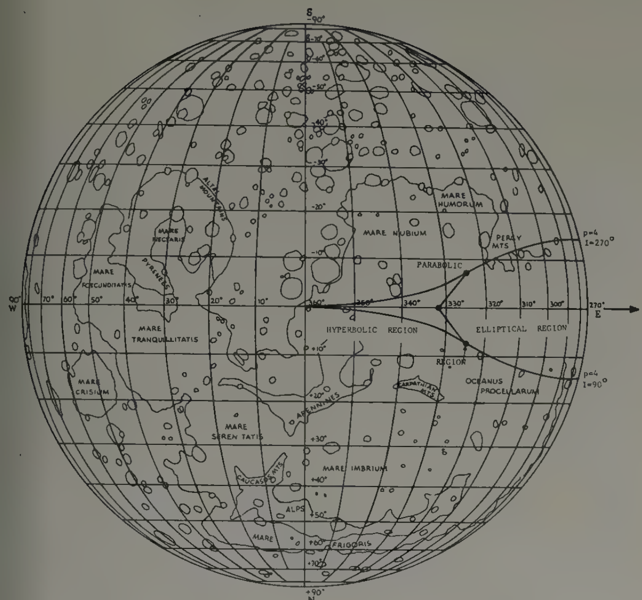

Fig. 6. Normal lunar impact for direct orbits

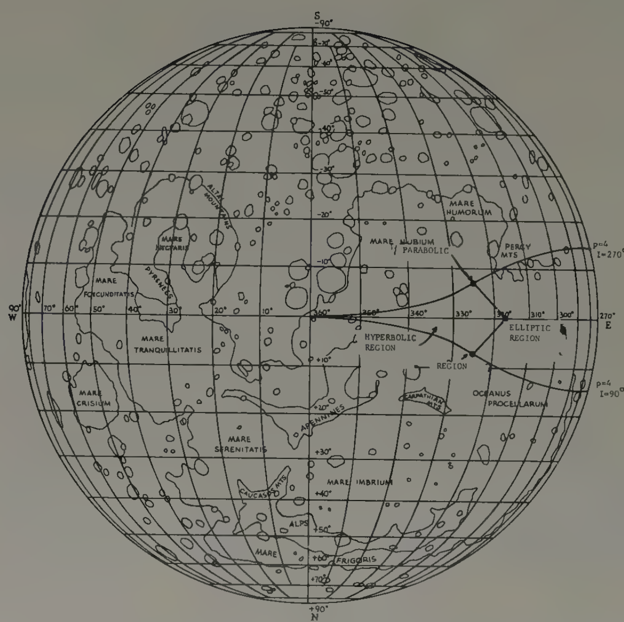

Fig. 7. Normal lunar impact for retrograde orbits

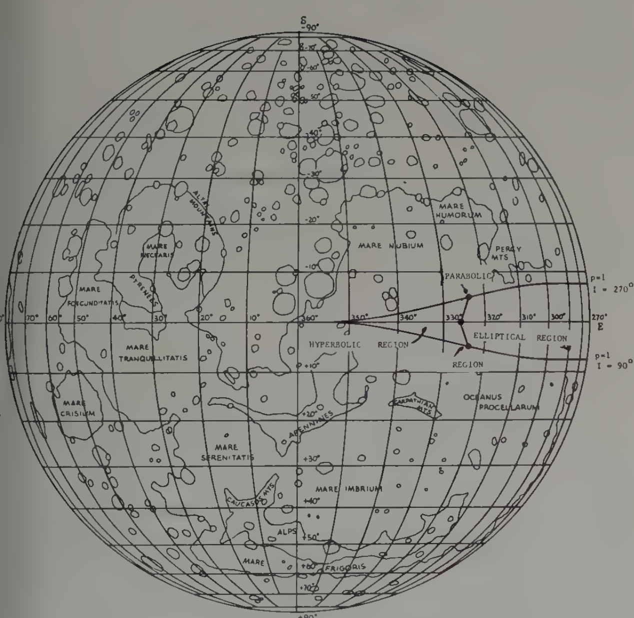

Fig. 8. Normal lunar impact for direct orbits

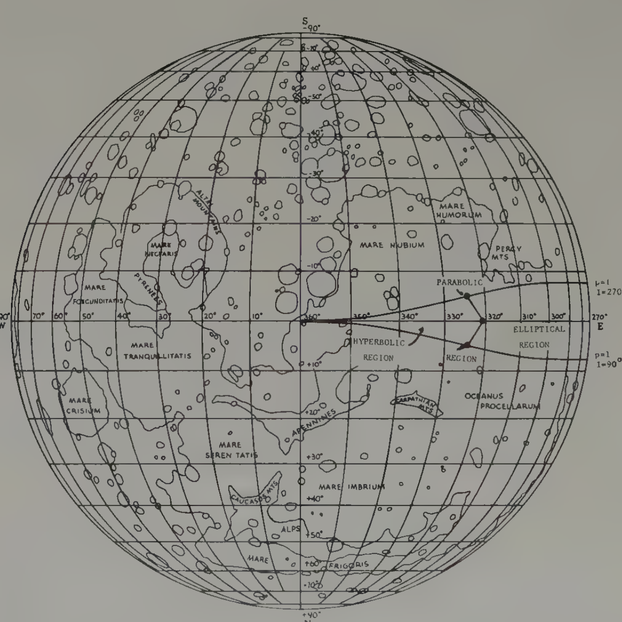

Fig. 9. Normal lunar impact for retrograde orbits

Error Analysis of Satellite Orbits in the Presence of Drag

Frederick V. Pohle*

Abstract

First order perturbation solutions of satellite orbits in the presence of drag were used to determine the error in the altitude, at the end of a half revolution, due separately to (i) errors in the initial altitude, (ii) initial horizontal speed, (iii) perturbation parameter, and (iv) decay parameter in the exponential density function. The error in the speed, at the end of a half revolution, was also determined as a function of the same parameters as in (i) to (iv). The present analysis is restricted to the case of a horizontal velocity at the initial position. In the absence of drag, information can be obtained easily on the effect of initial non-zero vertical speed upon the orbit.

I: Introduction

The trajectory analysis of a manned satellite may be divided into three stages: (i) take-off, (ii) orbital phase, and (iii) re-entry. This paper is concerned only with (ii), namely, it is assumed that certain conditions are prescribed at the end of (i) which serve as the initial conditions for (ii) and that these are sufficient to allow the satellite to stay in orbit for some time before the re-entry phase occurs.

Great accuracy in guidance is required to place the satellite in a given orbit. Inevitably there will be deviations (errors) from the assumed initial conditions. It is important to be able to determine the effects of these errors upon quantities of physical interest. In particular, the initial altitude, initial speed, perturbation parameter (which involves mass, drag coefficient and area of the satellite), and the decay constant in the assumed exponential density, may all vary from assumed or given values. This paper is concerned with the determination of the effect of these errors upon the altitude and speed at the end of a half revolution.

A precise orbital analysis would require the consideration of many effects. However, in phase (ii) the effect of drag is most pronounced and simplifying assumptions can be made; the precise assumptions are stated under Equations of Motion. A more complete discussion is contained in [3] and the re-entry phase i discussed, for example, in [2]. The present work is a extension of the work of Roberson, [1] who obtaine first-order drag corrections.

Roberson, [1], was concerned only with the satellit life-time study. Rather than compute each circui numerically on a digital computer, each complet circuit was computed analytically and the termina conditions of one circuit were used as the initial condi tions for the next circuit. In effect the orbital analysi was based upon difference equations in the revolutio number and only the difference equations were evalu ated on the computer. This approach to the study o satellite life-times is a significant improvement ove purely numerical studies. In the present problem it i of more importance to analyze each half-revolutio near re-entry and this has been done.

For completeness, the method of solution is out lined and in particular all quantities which occur in th final calculations are fully defined. The error deriva tives have been calculated from the first-order dra solutions under the uniform assumption of a zero initia vertical component of velocity. Since an error analysi for the vertical speed is also of value, a simplifie analysis is presented for this case. The present analysi must be modified if a drag correction is also to be ob tained in this case, and is not attempted in this report

Finally, some additional remarks have been mad on the possible presence of lift, second-order drag ef fects, non-zero initial vertical speed, and other modifi cations in the analysis.

II: Equations of Motion

The earth is assumed to be a sphere and the atmos phere is assumed to be non-rotating. Under these con ditions the orbit will be in a plane. If the drag is tan gential and proportional to the local density and to th square of the speed then the equations of motion i polar coordinates (r, β) are:

$$d^2r/dt^2 - r(d\beta/dt)^2 + K/r^2 = -(C_DA/2m)\rho(h)\dot{r}[(\dot{r})^2 + (r\dot{\beta})^2]^{1/2} \quad (1$$

$$r(d^2\beta/dt^2) + 2(dr/dt)(d\beta/dt) = -(C_DA/2m)\rho(h)r\dot{\beta}[(\dot{r})^2 + (r\dot{\beta})^2]^{1/2} \quad (2$$

It will be convenient to introduce the appropriat

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The results in this paper were obtained as part of an overall study of flight instrumentation conducted by the Bell Aircraft Corporation under Contract No. AF 33(616)-5943 for the Air Force, and carried out by the General Applied Science Laboratories, Inc.

non-dimensional coordinates as

$$\xi = a/r \tag{3}$$

$$\eta = Ka/(r^2\dot{\beta})^2 \tag{4}$$

The Equations (1, 2) contain $r(t)$ and $\beta(t)$: through (3, 4) the variables will be changed to $\xi(\beta)$ and $\eta(\beta)$ and primes ($'$) will be used to denote differentiation with respect to β. If the substitutions (3, 4) are carried out in (1, 2) the resulting equations, following Roberson [1], can be written as

$$\xi'' + \xi = \eta \tag{5}$$

$$\eta'/\eta = (\nu\rho/\xi)[1 + (\xi'/\xi)^2]^{1/2} \tag{6}$$

Equations (5, 6) are the equations used in the subsequent analysis.

III: Perturbation Solution

It is assumed that expansions can be written in the form

$$\xi(\beta) = \xi^{(0)} + (\nu\rho_0)\xi^{(1)} + \cdots \tag{7}$$

$$\eta(\beta) = \eta^{(0)} + (\nu\rho_0)\eta^{(1)} + \cdots \tag{8}$$

where the non-dimensional parameter $(\nu\rho_0)$ is assumed to be small compared with unity; only first order corrections will be computed and the zero-order terms are naturally the Kepler solutions which define the reference (drag-free) orbit. The zero-order equations are:

$$\xi^{(0)''} + \xi^{(0)} = \eta^{(0)} \tag{9}$$

$$\eta^{(0)'} = 0 \tag{10}$$

and it follows that $\eta^{(0)}$ is a constant, which becomes the right side of (9).

As initial conditions it will be assumed that at $t = 0$, $\beta = 0$ and that $r = r_1(\xi = \xi_1)$; $V_r = 0(\xi' = 0)$ and $\eta_1 = (Ka/r_1^2V_\beta^2)$ where V_β is the given speed. The solutions to equations (9, 10) can then be written as

$$\eta^{(0)}(\beta) = \eta_1 = \text{constant} \tag{11}$$

$$\xi^{(0)}(\beta) = \eta_1[1 + \epsilon_1 \cos \beta] \tag{12}$$

where

$$\epsilon_1 = (\xi_1/\eta_1) - 1 \tag{13}$$

The eccentricity of the orbit is given by (13), and (11) expresses the fact that the area is swept out uniformly in time. This Kepler solution is used in the first-order drag solution. The equations for the correction are:

$$\xi^{(1)''} + \xi^{(1)} = \eta^{(1)} \tag{14a}$$

$$\eta^{(1)'}(\beta) = \frac{\eta^{(0)}(\beta)\rho(\xi^{(0)})}{\rho_0\xi^{(0)}}\left[1 + \left(\frac{\xi^{(0)'}(\beta)}{\xi^{(0)}(\beta)}\right)^2\right]^{1/2} \tag{14b}$$

and

$$\eta^{(1)} = (1/\rho_0)\int_0^\beta \rho[\eta_1(1 + \epsilon_1 \cos \beta)] \times \left\{\frac{(1 + 2\epsilon_1 \cos \beta + \epsilon_1^2)^{1/2}}{(1 + \epsilon_1 \cos \beta)^2}\right\} d\beta \tag{15}$$

Equation (15) follows from (14b) and (12). It is necessary to expand the terms which do not involve the density, as follows:

$$\frac{(1 + 2\epsilon_1 \cos \beta + \epsilon_1^2)^{1/2}}{(1 + \epsilon_1 \cos \beta)^2} = \sum_{n=0}^{\infty} \varphi_n(\epsilon_1) \cos n\beta \tag{16}$$

The coefficients $\varphi_n(\epsilon_1)$ are defined in the Notation and a sufficient number of terms are shown to make possible solutions which systematically contain terms up to the third order in the eccentricity. It should be pointed out that the third coefficient of $\varphi_0(\epsilon_1)$ in [1] is given incorrectly.

The density function in (15) can be written as

$$\rho[\eta_1(1 + \epsilon_1 \cos \beta)] = \rho(\eta_1) \exp(\lambda \cos \beta) \tag{17}$$

where

$$\lambda = \eta_1\alpha_1\epsilon_1 \tag{18}$$

and $\rho(\eta_1)$ is defined under Notation; clearly, at $\beta = (\frac{1}{2})\pi$ the exponential term in (17) is unity and the density $\rho(\eta_1)$ is that at the altitude mid-way (in angular measure) between perigee and apogee. It is stated in [1] that the value of λ is about 25 for $\epsilon_1 = 0.2$. A value of the eccentricity of $\frac{1}{20}$ would correspond to an altitude change of about 400 miles; in this case λ would be about 6.25. The only density which occurs in the final calculations is $\rho(\eta_1)$; the actual densities vary from this value by the factors e^λ and $e^{-\lambda}$.

From [4], page 164, No. 103, it is known that

$$\frac{1}{\pi}\int_0^\pi \exp(\lambda \cos \beta)\cos n\beta \, d\beta = \frac{2}{\pi}\int_0^{2\pi} \exp(\lambda \cos \beta) \cos n\beta \, d\beta = I_n(\lambda) \tag{19}$$

The result (19) makes it possible to write the results for the half-revolution as well as for the full-revolution and attention will be restricted to the former case. The result for (15) can be written as $(\beta = \pi)$

$$\eta^{(1)}(\pi) = [\pi\rho(\eta_1)/\rho_0] \sum_{n=0}^{\infty} \varphi_n I_n \tag{20}$$

where the argument of φ_n is ϵ_1 and that of I_n is λ.

Since the right hand side of (14a) is known the solution can be written immediately in the integral form

$$\xi^{(1)}(\beta) = [\rho(\eta_1)/\rho_0] \sum_{n=0}^{\infty} \varphi_n(\epsilon_1) \times \int_0^\beta \exp(\lambda \cos \beta) \cos ny \, [\cos(\beta - y) - 1] \, dy \tag{21}$$

and therefore

$$\xi^{(1)}(\pi) = -[\pi\rho(\eta_1)/2\rho_0] \sum_{n=0}^{\infty} \varphi_n[I_{n+1} + 2I_n + I_{n-1}] \quad (22)$$

or

$$\xi^{(1)}(\pi) = [\pi\rho(\eta_1)/\rho_0] \sum_{n=0}^{\infty} \varphi_n B_n \quad (23)$$

where

$$B_n = B_n(\lambda) = -(\tfrac{1}{2})(I_{n+1} + 2I_n + I_{n-1}) \quad \text{and} \quad B_n = B_{-n}$$

The first order corrections are given by (20) and (23). In the following it will be necessary to evaluate I_n' and B_n'. From [4] (page 163, No. 90), it is known that

$$I_n' = (\tfrac{1}{2})(I_{n-1} + I_{n+1}) \quad (24)$$

If the definition of B_n is used, together with (24) the result is that

$$B_n' = (\tfrac{1}{2})(B_{n-1} + B_{n+1}) \quad (25)$$

which is the same functional form as (24). Thus (20) and (23) have the same form and the results for one will be the same as for the other in terms of B_n and I_n. The differentiations are taken with respect to the variable λ.

It is now possible to write down the results for ξ and η at $\beta = \pi$, as follows:

$$\xi(\pi) = (2\eta_1 - \xi_1) + \pi\nu\rho(\eta_1) \sum_0^{\infty} \varphi_n B_n \quad (26)$$

$$\eta(\pi) = \eta_1 + \pi\nu\rho(\eta_1) \sum_0^{\infty} \varphi_n I_n \quad (27)$$

where, for convenience, it is to be recalled that

$$\begin{aligned}
\xi_1 &= (a/r_1) \\
\eta_1 &= (ga/V_\beta^2)\xi_1^2 = \xi_1^2\zeta_1 \\
\zeta_1 &= (ga/V_\beta^2) \qquad\qquad (28) \\
\epsilon_1 &= [(\xi_1/\eta_1) - 1]; \qquad \varphi_n = \varphi_n(\epsilon_1) \\
\lambda &= \eta_1\alpha_1\epsilon_1; \qquad I_n = I_n(\lambda); \qquad B_n = B_n(\lambda)
\end{aligned}$$

The first-order solutions (26, 27) will be used together with the definitions listed in (28) to determine the error derivatives.

IV: Error Derivatives

The quantities which will be varied are ξ_1, ζ_1, α_1, ν. If the first alone is varied, for example, this implies an error in the initial altitude. The derivatives which are needed in the final calculations are

$$\begin{aligned}
\partial\eta_1/\partial\xi_1 &= 2\eta_1/\xi_1 \\
\partial\epsilon_1/\partial\xi_1 &= -1/\eta_1 \\
\partial\lambda/\partial\xi_1 &= \alpha_1[1 - (2\eta_1/\xi_1)] \qquad (29) \\
\partial\rho(\eta_1)/\partial\xi_1 &= (2\alpha_1\eta_1/\xi_1)\rho(\eta_1)
\end{aligned}$$

$$\begin{aligned}
\partial\eta_1/\partial\zeta_1 &= \xi_1^2 \\
\partial\epsilon_1/\partial\zeta_1 &= -\xi_1^3/\eta_1^2 \qquad (30 \\
\partial\lambda/\partial\zeta_1 &= \alpha_1\xi_1^2 \\
\partial\rho(\eta_1)/\partial\zeta_1 &= -\alpha_1\xi_1^2\rho(\eta_1)
\end{aligned}$$

$$(\partial\zeta_1/\partial V_\beta) = -2ga/V_\beta^3 \qquad (31$$

$$\begin{aligned}
\partial\rho(\eta_1)/\partial\alpha_1 &= \eta_1\rho(\eta_1) \qquad (32 \\
\partial\lambda/\partial\alpha_1 &= \eta_1\epsilon_1 = (\xi_1 - \eta_1)
\end{aligned}$$

The rate of change of final altitude with respect t the initial altitude can be written as

$$\frac{\partial\xi(\pi)}{\partial\xi_1} = [4(\eta_1/\xi_1) - 1] + \pi\nu\rho(\eta_1) \sum_0^{\infty} [\varphi_n B_n' \alpha_1[1 - 2(\eta_1/\xi_1)] + \varphi_n' B_n(-1/\eta_1) + \pi\nu\rho(\eta_1)\, \alpha_1(2\eta_1/\xi_1) \sum_0^{\infty} \varphi_n B$$

where

$$\varphi_n' = d\varphi_n/d\epsilon_1 = \Phi_n(\epsilon_1)$$

$$B_n' = dB_n/d\lambda = (\tfrac{1}{2})[B_{n-1} + B_{n+1}]$$

$$\begin{aligned}
\frac{\partial\xi(\pi)}{\partial\xi_1} &= [4(\eta_1/\xi_1) - 1] \\
&+ \pi\nu\rho(\eta_1) \sum_{n=0}^{\infty} \Big\{\Phi_n B_n(-1/\eta_1) + (2\alpha_1\eta_1/\xi_1)\varphi_n B_n \qquad (33 \\
&+ \frac{\alpha_1}{2}[1 - 2(\eta_1/\xi_1)]\varphi_n(B_{n-1} + B_{n+1})\Big\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\xi(\pi)}{\partial\zeta_1} &= 2\xi_1^2 + \nu\pi\rho(\eta_1) \sum_{n=0}^{\infty} [\varphi_n B_n'(-\alpha_1\xi_1^2) \\
&\quad + \varphi_n' B_n(-\xi_1^3/\eta_1^2)] \\
&\quad + (\nu\pi)\,\rho(\eta_1)\,\alpha_1\xi_1^2 \sum_{n=0}^{\infty} (\varphi_n B_n) \qquad (34 \\
\frac{\partial\xi(\pi)}{\partial\zeta_1} &= 2\xi_1^2 + \nu\pi\rho(\eta_1) \sum_{n=0}^{\infty} \\
&\quad \cdot\Big\{-\xi_1^3/\eta_1^2\,\Phi_n B_n + \alpha_1\xi_1^2\varphi_n B_n \\
&\quad - (\alpha_1\xi_1^2/2)(B_{n-1} + B_{n+1})\Big\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\xi(\pi)}{\partial\alpha_1} &= \pi\nu\rho(\eta_1) \sum_{n=0}^{\infty} \varphi_n B_n'(\xi_1 - \eta_1) \\
&\quad + \pi\nu\eta_1\rho(\eta_1) \sum_{n=0}^{\infty} \varphi_n B_n \qquad (35 \\
&= \nu\pi\rho(\eta_1) \sum_{n=0}^{\infty} \{(\xi_1 - \eta_1)(1/2) \\
&\quad \varphi_n[B_{n-1} + B_{n+1}] + \eta_1\varphi_n B_n\}
\end{aligned}$$

$$\frac{\partial\xi(\pi)}{\partial\nu} = \pi\rho(\eta_1)\sum_{n=0}^{\infty}\varphi_n B_n \qquad (36)$$

$$\nu = (C_D A a/m)$$

nd changes in the values C_D, A and M can be related o changes in ν. The derivatives which show the effect f different quantities upon the altitude are given by 33) to (36).

The corresponding derivatives for the speed can be vritten down directly when it is recalled that (26) and 27) are essentially the same in the drag terms.

Let

$$(\pi) = \eta(\pi)/\xi^2(\pi)$$

$$1/\zeta(\pi)]\partial\zeta(\pi)/\partial\xi_1 = [1/\eta(\pi)]\partial\eta(\pi)/\partial\xi_1 - [2/\xi(\pi)]\partial\xi(\pi)/\partial\xi_1 \qquad (37)$$

$$1/\zeta(\pi)]\partial\zeta(\pi)/\partial\zeta_1 = [1/\eta(\pi)]\partial\eta(\pi)/\partial\zeta_1 - [2/\xi(\pi)]\partial\xi(\pi)/\partial\zeta_1$$

nd

$$1/\zeta(\pi)]\partial\zeta(\pi)/\partial\alpha_1 = [1/\eta(\pi)]\partial\eta(\pi)/\partial\alpha_1 - [2/\xi(\pi)]\partial\xi(\pi)/\partial\alpha_1$$

$$1/\zeta(\pi)]\partial\zeta(\pi)/\partial\nu = [1/\eta(\pi)]\partial\eta(\pi)/\partial\nu - [2/\xi(\pi)]\partial\xi(\pi)/\partial\nu \qquad (38)$$

The definition (37) can be used to determine the erivatives which have been written in (38). The erivatives in the second terms on the right hand sides f (38) are known from equations (33) to (36). The erivatives in the first terms can be obtained from an xamination of (27) which shows that the drag terms ill be the same as those of (26) if B_n is replaced by $_n$; therefore the only term which is left is the first erm of (27) which is the constant term from the eference orbit solution. The derivatives of the first erms are known from the first results of (29) and 30), respectively, and therefore the derivatives for ne speed as a function of initial altitude, initial speed, rag parameter, and exponential decay constant are so known.

Examination of the leading term of (33) shows that, the case of a circular orbit, the drag-free term is 3, at is, an error in initial altitude is multiplied by 3 the end of a half revolution. This can also be proved rectly and it is easy to understand the result when is recalled that the speed which is appropriate to a ven altitude for a circular orbit will cause the satel- e to move in an ellipse of small eccentricity if the eed is kept constant as the altitude is changed ghtly.

The present analysis must be substantially modified initial vertical speed is to be added as a parameter. ne presence of such a parameter causes the apse line the reference ellipse to be rotated and to introduce ditional terms which will not modify the analysis if ull revolution is considered but which will change the analysis completely if only a half-revolution is considered. If the analysis is based only on a drag-free orbit and if the initial speed is given through the value ξ_1' then the equation of the orbit is

$$\xi(\beta) = \eta_1 + [(\xi_1 - \eta_1)]\cos\beta + \xi_1'\sin\beta$$

and the eccentricity of the orbit is given by

$$\{[(\xi_1/\eta_1) - 1]^2 + (\xi_1'/\eta_1)^2\}^{1/2}$$

If, for example, the vertical speed was initially zero, so that the orbit would be circular, an error in this value would cause the orbit to be slightly elliptical and the errors in altitude, for example, from this case alone could become appreciable.

V: Concluding Remarks

The present analysis has neglected certain effects which may also be important. For example, the presence of lift in the governing equations would seriously modify the results; this additional force can be included but the analysis must be modified. If the lift-to-drag ratio is small then a reasonable approximate solution is again possible.

In the work of [1] the density is assumed to be small enough for the quadratic approximation to be negligibly small. In the present case where the flight is at lower altitudes the second-order correction could become significant. The higher terms can be included systematically in principle but the practical difficulties are such that a modified analysis would have to be used.

Finally, other basic equations could be used with profit in the analysis. In particular other dynamical variables can be used and different perturbation parameters can be introduced to fit special circumstances. Energy relationships can also play a significant role in the determination of the variation of orbital elements.

Notation and Definitions

a = radius of the spherical earth (ft)

A = cross-sectional area of the body, normal to the flow field and assumed constant (ft^2)

B_n = $-(\frac{1}{2})(I_{n-1} + 2I_n + I_{n+1})$; $B_{-n} = B_n(n \geq 0)$ $B_n = B_n(\lambda)$

B_n' = $(\frac{1}{2})(B_{n-1} + B_{n+1})(n \geq 0)$ $B_n' = dB_n/d\lambda$

C_D = drag coefficient (constant)

g = acceleration due to gravity at surface of the earth (ft/sec^2)

h = altitude above the surface of the earth $= r - \sigma$ (ft)

I_n = $I_n(\lambda)$ modified Bessel function of the first kind; $\lambda = \alpha_1\eta_1\epsilon_1$; $I_{-n} = I_n(n \geq 0)$

I_n' = $(\frac{1}{2})(I_{n-1} + I_{n+1})(n \geq 0)$ $I_n' = dI_n/d\lambda$

K = ga^2 (ft^3/sec^2)

m = mass (slugs)

n = integer, 0, 1, 2, $\cdots$

r = distance from the center of the earth to the orbit (ft)

r_1 = initial value of r $(t = 0)$ (ft)

t = time (sec)

V_β = velocity component perpendicular to the radius vector r (positive in the sense of increasing β) (ft/sec)

V_r = velocity component parallel to the radius vector r (positive outward) (ft/sec)

$\cdot$ = d/dt; differentiation with respect to the time t

α_1 = altitude parameter for the reference orbit

β = angular advance of the body from a fixed reference line (inertial axis). The location $\beta = 0$ corresponds either to perigee or to apogee, that is, it is at an apse. (radians)

δ_1 = constant in the exponential density function

ϵ_1 = eccentricity of the reference (drag-free) orbit $= (\xi_1/\eta_1) - 1$

ξ = a/r; non-dimensional measure of the altitude. The quantity is always less than unity and a change of one part in 4000 is equal to a change in altitude of one mile.

ξ_1 = a/r_1 initial value of ξ.

$\xi^{(0)}$ = drag-free value of ξ, as a function of β

$\xi^{(1)}$ = first-order drag correction to ξ, that is, $\xi = \xi(\beta) = \xi^{(0)} + (\nu\rho_0)\xi^{(1)}$

η = $(ga/V_\beta^2)\xi^2$

η_1 = initial value of η

$\eta^{(0)}$ = drag-free value of η

$\eta^{(1)}$ = first-order drag correction to η, that is, $\eta = \eta(\beta) = \eta^{(0)} + \nu\rho_0\eta^{(1)}$

ζ = ga/V_β^2 non-dimensional measure of V_β

ζ_1 = initial value of ζ

$\rho(h)$ = density at the altitude h. Typical values are: (slugs/ft^3)

h(feet)	(slugs ft^3)
300,000	6×10^{-9}
400,000	6.5×10^{-11}
500,000	3×10^{-12}
600,000	4.5×10^{-13}
700,000	1.3×10^{-13}

The atmosphere is assumed as locally e ponential, that is $\rho = \exp(\alpha\xi + \delta)$ whe α and δ are known constants and exp ($= e^x$.

ρ_0 = density at the initial altitude $(r_1 - a)$ (slugs/ft

$\rho(\eta_1)$ = reference density at the altitude correspon ing to η, that is the altitude $= (a/\eta_1) -$ This altitude is intermediate in value the maximum and minimum altitudes. (slugs/ft

φ_0 = $1 + (\frac{3}{4})(\epsilon_1)^2 + (\frac{45}{64})(\epsilon_1)^4 + \cdots$

φ_1 = $-(\epsilon_1) - (\frac{9}{8})(\epsilon_1)^3 - (\frac{3}{32})(\epsilon_1)^5 + \cdots$

φ_2 = $(\frac{1}{4})(\epsilon_1)^2 + (\frac{9}{16})(\epsilon_1)^4 + \cdots$

φ_3 = $-(\frac{1}{8})(\epsilon_1)^3 + \cdots$

φ_4 = $-(\frac{17}{64})(\epsilon_1)^4 + \cdots$

Φ_0 = $(d\varphi_0/d\epsilon_1) = (\frac{3}{2})(\epsilon_1) + (\frac{45}{16})(\epsilon_1)^3 + \cdots$

Φ_1 = $(d\varphi_1/d\epsilon_1) = -1 - (\frac{27}{8})(\epsilon_1)^2 - (\frac{15}{32})(\epsilon_1$ $+ \cdots$

Φ_2 = $(d\varphi_2/d\epsilon_1) = -(\frac{1}{2})(\epsilon_1) + (\frac{9}{4})(\epsilon_1)^3 + \cdots$

Φ_3 = $(d\varphi_3/d\epsilon_1) = -(\frac{3}{8})(\epsilon_1)^2 + \cdots$

Φ_4 = $(d\varphi_4/d\epsilon_1) = -(\frac{17}{16})(\epsilon_1)^3 + \cdots$

ν = C_DAa/m; drag parameter (ft^3/slugs

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Technical Notes

The Intersection of Coplanar, Confocal Conic Sections

Geza S. Gedeon*

bstract

A simple method is presented to calculate the intersections of two coplanar confocal orbits.

iscussion

Given two coplanar position and velocity vectors the uestion arises: where will their orbit intersect? In the follow-

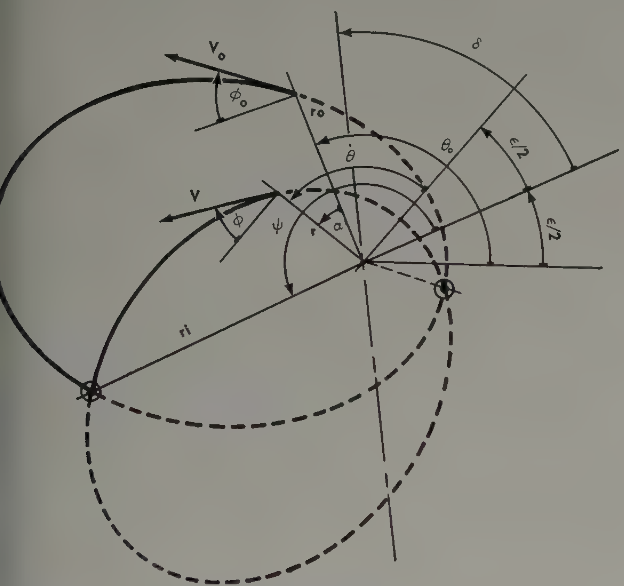

FIG. 1. Notation

g a simple method is presented to calculate the intersections these orbits. Using the notations defined on Fig. 1 and in e Nomenclature the semi-latus rectums and eccentricities the two orbits can be calculated from the following equa-ons:

$$p = rK^2 \cos^2 \phi \tag{1}$$

$$a = \frac{r}{2 - K^2} \tag{2}$$

$$e = \sqrt{1 - p/a} \tag{3}$$

or the derivation of these equations see [1]. The true anom-ies related to the given radii are obtained from the equation Ref. [2]

$$\theta = \phi + \tan^{-1}\left[\frac{\tan \phi}{K^2 - 1}\right] \tag{4}$$

om Fig. 1 it can be seen that the skew angle between the line of apsides is

$$\epsilon = \theta_0 - (\theta - \alpha) \tag{5}$$

If a new coordinate is chosen to bisect the angle ϵ, then the common radii of intersection can be expressed as

$$\frac{p_0}{1 + e_0 \cos\left(\psi + \frac{\epsilon}{2}\right)} = \frac{p}{1 + e \cos\left(\psi - \frac{\epsilon}{2}\right)} \tag{6}$$

Using simple trigonometric relations Eq. 6 can be rewritten as

$$\left[\frac{e}{p} - \frac{e_0}{p}\right] \cos\frac{\epsilon}{2} \cos\psi + \left[\frac{e}{p} + \frac{e_0}{p}\right] \sin\frac{\epsilon}{2} \sin\psi = \frac{1}{p_0} - \frac{1}{p} \tag{7}$$

If an angle δ is introduced by the definition

$$\delta = \tan^{-1}\left[\frac{\tan\frac{\epsilon}{2}}{\xi}\right] \tag{8}$$

where

$$\xi = \frac{\dfrac{p_0/p}{e_0/e} - 1}{\dfrac{p_0/p}{e_0/e} + 1} \tag{9}$$

then Eq. 7 can be expressed as

$$\psi = \delta + \cos^{-1}\left[-\frac{\zeta}{e_0}\frac{\cos\delta}{\cos\frac{\epsilon}{2}}\right] \tag{10}$$

with

$$\zeta = \frac{p_0/p - 1}{\dfrac{p_0/p}{e_0/e} - 1} \tag{11}$$

Figure 2 presents the intersection factors ξ and ζ as functions of the latus rectum and eccentricity ratios.

In Eq. 8 the arc tangent has two values. If these values are measured from the polar axis of the chosen coordinate system, a full line is obtained and Eq. 10 yields two symmetrical solutions with respect to this δ axis.

The true anomalies of the intersection related to the line of apsides of the intersecting conic sections can be obtained by adding resp. subtracting $\epsilon/2$ from ψ.

Eq 10 can be used to determine the conditions when the conics intersect, are cotangent, or one is wholly contained in the other. For this purpose Eq 10 will be rewritten by Eq 8 in the following form:

$$\sec^2(\psi - \delta) = \frac{e_0^2}{\zeta^2}\cos^2\frac{\epsilon}{2}\left[1 + \frac{\tan^2\epsilon/2}{\xi^2}\right], \tag{12}$$

or

$$\sec^2(\psi - \delta) = \frac{e_0^2}{\zeta^2}\left[\frac{1}{\xi^2} + \left(1 - \frac{1}{\xi^2}\right)\frac{1 + \cos\epsilon}{2}\right]. \tag{13}$$

If ξ and ζ are resubstituted then Eq 13, after a little rear-

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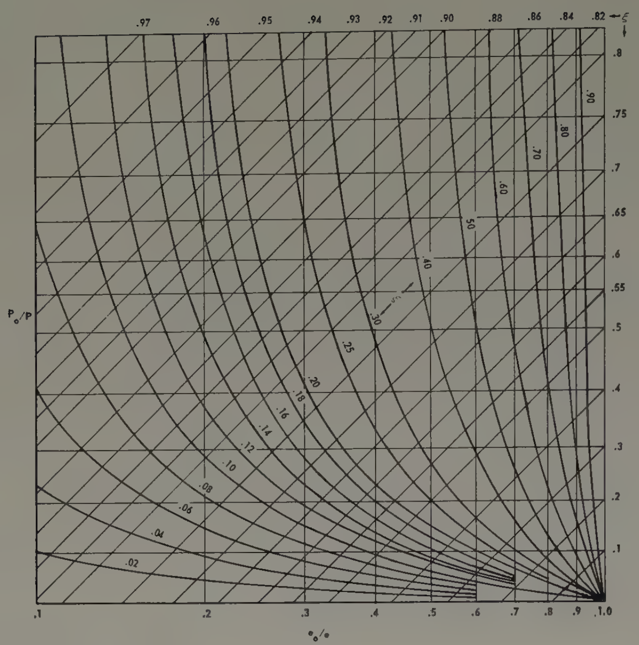

FIG. 2. Intersection factors ξ and ζ as functions of latus rectum ratio and eccentricity ratio.

rangement, becomes

$$\sec^2(\psi - \delta) = \frac{\left(\dfrac{e}{p}\right)^2 + \left(\dfrac{e_0}{p_0}\right)^2 - 2\dfrac{e}{p}\dfrac{e_0}{p_0}\cos\epsilon}{\left(\dfrac{1}{p} - \dfrac{1}{p_0}\right)^2} \tag{14}$$

Since the secant must be greater than one, the numerator must be greater than the denominator for intersection and equal for cotangency as it has been already shown by Lawden [3] through geometrical argument. To give a physical explanation to Eq 14 the following steps will be taken

$$\tan^2(\psi - \delta) = \sec^2(\psi - \delta) - 1$$

$$= \frac{-\left[\dfrac{1 - e^2}{p^2} + \dfrac{1 - e_0^2}{p_0^2} + 2\dfrac{e}{p}\dfrac{e_0}{p_0}\cos\epsilon - \dfrac{2}{pp_0}\right]}{\dfrac{1}{p^2} - \dfrac{2}{pp_0} + \dfrac{1}{p_0^2}} \tag{15}$$

Both the numerator and the denominator will be then multiplied by aa_0pp_0 yielding the following

$$\tan^2(\psi - \delta) = \frac{-[a_0^2(1 - e_0^2) + a^2(1 - e^2) + 2(ae)(a_0e_0)\cos\epsilon - 2aa_0]}{aa_0\dfrac{(p_0 - p)^2}{pp_0}} \tag{16}$$

If the semi major axis is defined as positive for ellipse an negative for hyperbola then the linear eccentricity can b introduced with the same sign convention.

Thus Eq 16 becomes:

$$\tan(\psi - \delta) = \sqrt{\frac{c_0^2 - 2c_0c\cos\epsilon + c^2 - (a_0 - a)^2}{aa_0\dfrac{(p_0 - p)^2}{p_0p}}} \tag{17}$$

In this expression

$$c_0^2 - 2c_0c\cos\epsilon + c^2 = f^2 \tag{18}$$

f is the center spread.

To have real solution for Eq 17 the center spread must b greater or equal to the semi-major axis difference in case similar conics, and must be less than or equal to the sum the absolute values of the two semi-major axes if one of th conics is a hyperbola. In case of equality $\psi = \delta$ is the poi of cotangency.

Nomenclature

μ = $G(m_1 + m_2) \cong g_0R_0^2$ planetary constant
r = radius vector
V_0 = $\sqrt{\mu/r}$ = circular orbit velocity
V = orbital velocity
K = V/V_0 = Kepler Number
a = semi-major axis
c = linear eccentricity
e = c/a = numerical eccentricity
p = $(1 - e^2)a$ = semi-latus rectum
f = center spread
ϕ = angle of elevation
α = angle between two position vectors
θ = true anomaly measured from pericenter
ϵ = skew angle between the line of apsides
ψ = true anomaly measured from the bisector of ϵ
δ = axis of intersection symmetry measured from the b sector of ϵ
ξ = $\dfrac{\dfrac{p_0/p}{e_0/e} - 1}{\dfrac{p_0/p}{e_0/e} + 1}$ = first intersector factor
ζ = $\dfrac{p_0/p - 1}{\dfrac{p_0/p}{e_0/e} - 1}$ = second intersector factor

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Two Maneuver Ascents to Circular Orbits

J. F. Wolfe* and D. DeBra*

Characteristic velocity is a common criterion for evaluating cent and transfer paths [1]. It will be used in this note ough other criteria such as mass ratio may give different d sometimes more significant results [2].

Large circular orbits may be attained by coasting on a ansfer orbit after initial thrust and then performing a cond circularizing maneuver when the proper altitude is ached. The total propulsion requirements can be estimated evaluating the initial velocity at the start of the coast bit and adding the velocity impulse necessary to accomish the second maneuver. The sum is called v_T, the characristic velocity. Use of additional impulses may reduce v_T but also reduces reliability and requires additional staging for ckets not capable of multiple restarts. Therefore, only a ngle velocity impulse in addition to the initial velocity will considered.

The second maneuver will occur at apoapsis of the transfer bit, which must coincide with a point on the desired circur orbit. The usual methods for finding extrema show that is condition minimizes v_T with respect to the co-flight path ngle of the transfer orbit at the point of second maneuver. his condition for minimization is true for non-coplanar as ell as coplanar transfer orbits.

In this discussion quantities are normalized to the initial lius and circular speed at the initial radius, which automatically normalizes the constant of an assumed inverse-square-law, central-force field. Notation is defined in Fig. 1.

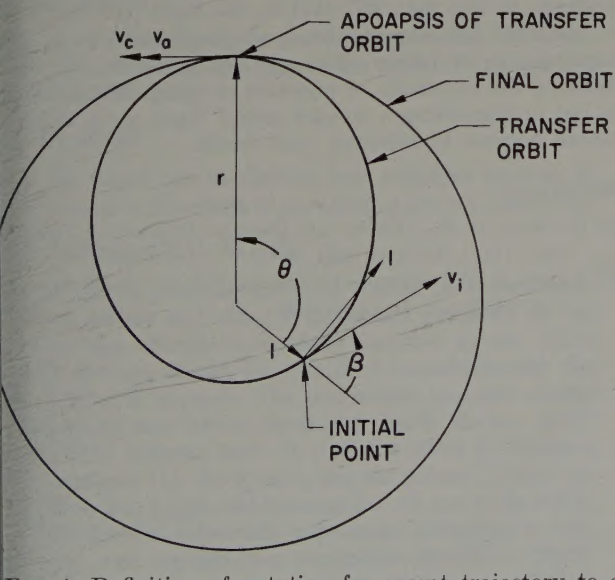

Fig. 1. Definition of notation for ascent trajectory to a cular orbit.

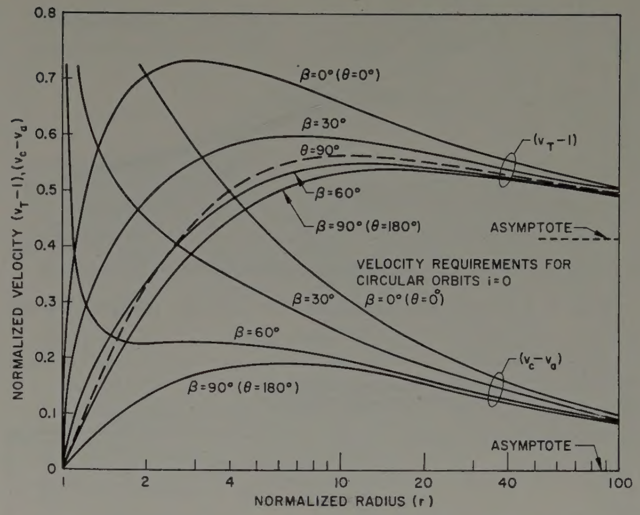

Fig. 2. Velocity requirements for circular orbits $i = 0$

The characteristic velocity for coplanar ascents is just a function of the radius of the desired circular orbit and one other parameter which defines what path was followed to get there, e.g., β, θ etc. The final velocity maneuvers $(v_c - v_a)$ and v_T are plotted in Fig. 2. It may seem surprising that v_T is not a monotonically increasing function of the final orbit radius, r. At small r the behavior is as expected, but for large r the final maneuver decreases more rapidly than the initial velocity increases thus causing a maximum in characteristic velocity as noted in [1].

The expressions for these quantities can be found parametrically in the co-flight path angle by using the conservation of momentum and energy along the transfer orbit.

$$v_a = v_i \sin \beta / r$$

$$v_i^2 = 2(1 - 1/r)/(1 - \sin^2\beta/r^2)$$

Circular speed is $(1/r)^{1/2}$ so the characteristic velocity is

$$\begin{aligned} v_T &= v_i + (v_c - v_a) = v_c + v_i(1 - v_a/v_i) \\ &= (1/r)^{1/2} + [2(1 - 1/r)(1 - \sin \beta/r)/(1 + \sin \beta/r]^{1/2} \end{aligned} \tag{1}$$

It is sometimes more convenient to express the results as a function of transfer angle θ. Since the angle θ is the supplement of the true anomaly of the initial point on the transfer orbit, the relation between β and θ is

$$\operatorname{ctn} \beta = \epsilon \sin \theta/(1 - \epsilon \cos \theta) \tag{2}$$

By solving the orbit equation for the semi-latus rectum and by equating the expression for this parameter at apoapsis to that at the initial point, one gets

$$\epsilon = (r - 1)/(r - \cos \theta)$$

and from (2)

$$\sin^2 \beta/r^2 = (1 - \cos \theta)/[2r^2 - (2r - 1)(1 + \cos \theta)]$$

which can be substituted in (1) to give the desired result.

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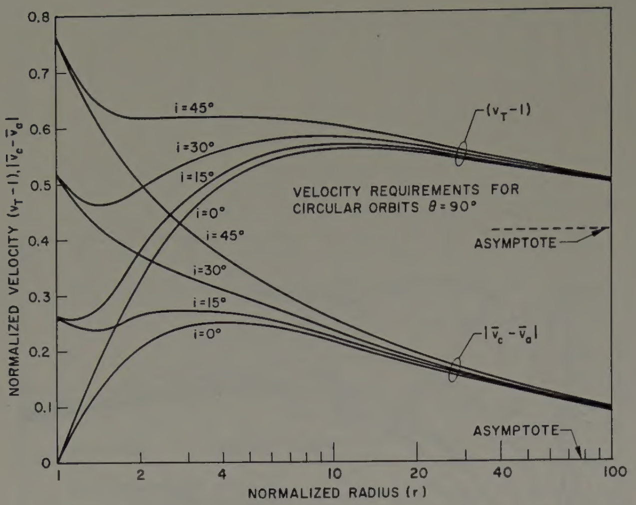

Fig. 3. Velocity requirements for circular orbits $\theta = 90°$

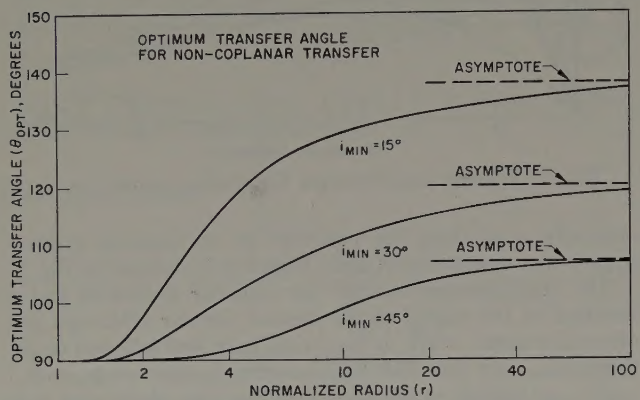

Fig. 4. Optimum transfer angle for non-coplanar transfer

When the desired orbit plane does not contain the initial point, the geometry is specified by r and the minimum inclination possible between the transfer orbit and the final orbit, $i_{\min}$. Again injection will occur at apoapsis of the transfer orbit. The second maneuver must also change the planes, so it must be the vector difference rather than the scalar difference plotted in Fig. 2. The characteristic velocity is then

$$v_T = v_i + |\bar{v}_c - \bar{v}_a| = v_i + (v_c^2 - 2v_c v_a \cos i + v_a^2)^{1/2}$$

where i is the inclination of the transfer orbit to the final orbit.

Choosing the transfer orbit $\theta = 90°$ results in the least inclination. This case is plotted in Fig. 3. For some i, for example 30°, extrema of both types occur.

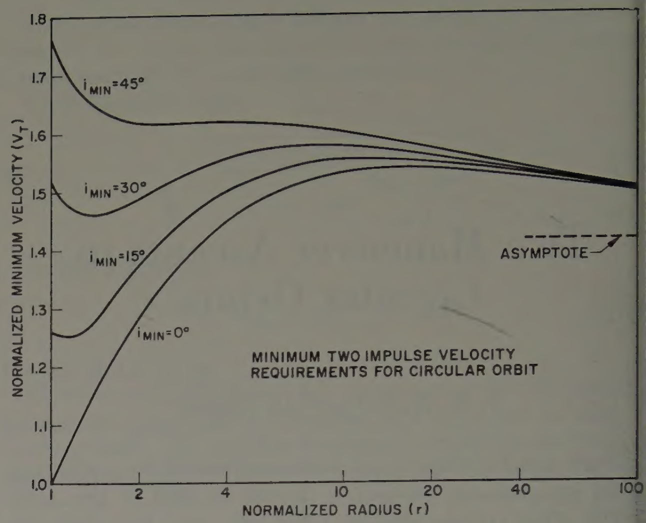

Fig. 5. Minimum two impulse velocity requirements fo circular orbits.

The minimum v_T does not occur for minimum inclination By including the geometrical relation

$$\sin i_{\min} = \sin i \sin \theta$$

v_T may be minimized.

Then for given r and $i_{\min}$, one additional parameter e.g. i, θ, β, etc. specifies the transfer orbit. The turning poin may be computed by setting the partial derivative of v_T wit respect to this parameter equal to zero. Here $\partial v_T/\partial\theta = 0$ wa used to compute θ_{opt}, the value of θ which minimizes v_T.

The derivative is algebraically complicated and not con venient to work with except for $r = 1$ and $r = \infty$. For $i_{\min} \neq$ 0 the value of θ_{opt} is 90° for $r = 1$ and as $r \rightarrow \infty$ it is give by the expression

$$\sin^2 \theta_{\text{opt}} = 2 \sin i_{\min} - \sin^2 i_{\min}$$

Other values of θ_{opt} are plotted as a function of r in Fig. 4 The corresponding values of v_T are given in Fig. 5.

In the non-coplanar case the optimum value of β wi generally be less than 90°; that is, the flight path angle wi be positive. The requirement for a large flight path angl may reduce propulsion efficiency during initial thrusting Relaxing the constraint of injection at apoapsis will increas v_T but in permitting a smaller initial flight path angle a overall increase in efficiency may result.

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The editors will appreciate the cooperation of authors in using the following directions for the preparation of manuscripts. These directions have been compiled with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

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Company Reports

The paper should not be merely a company report. If such a report is to be used as the basis for the paper, appropriate changes should be made in the title page. Lists of figures, tables of contents, and distribution lists should all be deleted.

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The title should be brief, but express adequately the subject of the paper. A footnote reference to the title should indicate any meeting at which the paper has been presented. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

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As far as possible, formulas should be typewritten. Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Each such symbol should be identified unambiguously the first time it appears. The distinction between capital and lower-case letters should be clearly shown. Avoid confusion between zero (0) and the letter *O*; between the numeral (1), the letter l, and the prime (′); between alpha and *a*, kappa and *k*, mu and *u*, nu and *v*, eta and *n*.

The level of subscripts, exponents, subscripts to subscripts, and exponents in exponents should be clearly indicated.

Complicated exponents and subscripts should be avoided when possible to represent by a special symbol.

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus

$$\frac{\cos(\pi x/2b)}{\cos(\pi\alpha/2b)}$$

is the preferred usage.

The intended grouping of handwritten formulas can be made clear by slight variations in spacing, but this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus

$(a + bx) \cos t$ is preferable to $\cos t\, (a + bx)$

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$$\{[a + (b + cx)^n] \cos ky\}^2$$

is required rather than $((a + (b + cx)^n) \cos ky)^2$.

Equations are numbered and referred to in text as (15).

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Bibliography

References should be grouped together in a bibliography at the end of the manuscript. References to the bibliography should be made by numerals between square brackets [4].

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for books—[1] HUNSAKER, J. C. and RIGHTMIRE, B. S., *Engineering Applications of Fluid Mechanics*, McGraw-Hill Book Co., New York, 1st ed., 1947, p. 397.

for periodicals—[2] Singer, S. F., "Artificial Modification of the Earth's Radiation Belt," *J. Astronaut. Sci.*, *6* (1959), 1–10.